1. Evaluate \( \iiint_D \vec{F} \cdot d\vec{S} \), where \( \vec{F} = (bxy^2, bx^2y, (x^2 + y^2)z^2) \) and \( S \) is the closed surface bounding the region \( D \) consisting of the solid cylinder \( x^2 + y^2 \leq a^2 \) and \( 0 \leq z \leq b \).

**Solution**

This is a problem for which the divergence theorem is ideally suited. Calculating the divergence of \( \vec{F} \), we get

\[
\nabla \cdot \vec{F} = (\partial_x, \partial_y, \partial_z) \cdot (bxy^2, bx^2y, (x^2 + y^2)z^2) = (x^2 + y^2)(b + 2z).
\]

Applying the divergence theorem we get

\[
\iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D (x^2 + y^2)(b + 2z) \, dV
\]

\[
= \int_0^{2\pi} \int_0^a \int_0^b \int_0^2 (r^2 + 2rz) \, r \, dz \, dr \, d\theta = 2b^2 \int_0^{2\pi} \int_0^a \int_0^2 (\frac{a^4}{4}) \, 2\pi = \pi a^4 b^2. \]

2. Find the flux of \( \vec{F} = (x, y^2, z) \) upward through the first-octant part \( S \) of the cylindrical surface \( x^2 + z^2 = a^2 \) for \( 0 \leq y \leq b \).

**Solution**

The surface \( S \) is one of five surfaces that form the boundary of the solid region \( D \) consisting of the part of the cylinder \( x^2 + z^2 = a^2 \) for \( 0 \leq y \leq b \) that lies within the first-octant. The other four surfaces are plane surfaces: \( S_1 \) lies in the plane \( z = 0 \), \( S_2 \) lies in the plane \( x = 0 \), \( S_3 \) lies in the plane \( y = 0 \), and \( S_4 \) lies in the plane \( y = b \). Orienting all surfaces so that the normal \( \vec{n} \) points outwards we get

\[
\vec{n}_1 = (0, 0, -1), \quad \vec{n}_2 = (-1, 0, 0), \quad \vec{n}_3 = (0, -1, 0), \quad \vec{n}_4 = (0, 1, 0).
\]

If we let \( S_{\text{tot}} \) represent the total boundary of \( D \), i.e. \( S_{\text{tot}} = S \cup S_1 \cup S_2 \cup S_3 \cup S_4 \), then we have

\[
\iiint_{S_{\text{tot}}} \vec{F} \cdot d\vec{S} = \iiint_S \vec{F} \cdot d\vec{S} + \iiint_{S_1} \vec{F} \cdot d\vec{S} + \iiint_{S_2} \vec{F} \cdot d\vec{S} + \iiint_{S_3} \vec{F} \cdot d\vec{S} + \iiint_{S_4} \vec{F} \cdot d\vec{S}
\]

\[
= \iiint_S \vec{F} \cdot d\vec{S} + \iiint_{S_1} (-z) dS + \iiint_{S_2} (-x) dS + \iiint_{S_3} (-y^2) dS + \iiint_{S_4} y^2 dS
\]

\[
= \iiint_S \vec{F} \cdot d\vec{S} + 0 + 0 + \iiint_{S_4} b^2 dS
\]

\[
= \iiint_S \vec{F} \cdot d\vec{S} + b^2 \text{(area of } S_4) = \iiint_S \vec{F} \cdot d\vec{S} + \frac{\pi a^2 b^2}{4}
\]

On the other hand, using the divergence theorem, we get

\[
\iiint_{S_{\text{tot}}} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D (2 + 2y) \, dV = 2V + 2V_y,
\]

1
where $V = \pi a^2 b/4$ is the volume of $D$, and $\bar{y} = b/2$ is the $y$-coordinate of the centroid of $D$. The final result is

$$\int \int_S \vec{F} \cdot \vec{n} \, dS = \int \int_{S_{\text{tot}}} \vec{F} \cdot \vec{n}_{\text{tot}} \, dS - \frac{\pi a^2 b^2}{4} = 2V + 2V \bar{y} - \frac{\pi a^2 b^2}{4} = \frac{\pi a^2 b}{2}. \quad \blacksquare$$

3. Using the divergence theorem, evaluate $\int \int_S (x^2 + y^2) \, dS$, where $S$ is the sphere $x^2 + y^2 + z^2 = a^2$.

\textbf{Solution}

On $S$ we have $\vec{n} = \frac{1}{a} \langle x, y, z \rangle$.

We would like to choose a vector field $\vec{F}$ so that $\vec{F} \cdot \vec{n} = x^2 + y^2$. Observe that $\vec{F} = \langle ax, ay, 0 \rangle$ will do.

If $D$ is the interior of the sphere $S$, then we have

$$\int \int_S (x^2 + y^2) \, dS = \int \int_D \vec{F} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D 2a \, dV = 2a \text{(volume of } D) = (2a)^2 \frac{4}{3} \pi a^3 = \frac{8}{3} \pi a^4. \quad \blacksquare$$

4. Calculate the flux of $\vec{F} = \langle x^3 + y \sin z, yz, zx^2 \rangle$ across the surface $S$, where $S$ is the boundary of the solid $D$ bounded by the hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = 1$, and the plane $z = 3$.

\textbf{Solution}

Using the divergence theorem we get:

$$\int \int_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D 3(x^2 + y^2 + 1) \, dV$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^{\sqrt{2}} (\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \left( \frac{2^{5/3} - 1}{3} \sin \phi + \frac{2^3 - 1}{3} \sin \phi \right) \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \left( \frac{31}{5} \sin \phi \right) \, d\phi \, d\theta$$

$$= 3 \int_0^{2\pi} \left[ \frac{31}{5} \left( \frac{2}{3} \sin \phi \right) + \frac{7}{3} \cos \phi \right]^{\pi/2}_0 \, d\theta = 3 \int_0^{2\pi} \left[ \frac{31}{5} \left( \frac{2}{3} \right) + \frac{7}{3} \right] \, d\theta = \frac{194\pi}{5}. \quad \blacksquare$$

5. Calculate the flux of $\vec{F} = \langle xy^2, yz, zx^2 \rangle$ across the surface $S$, where $S$ is the boundary of the solid $D$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$.

\textbf{Solution}

This is a problem for which the divergence theorem is ideally suited. Calculating the divergence of $\vec{F}$,
we get
\[ \nabla \cdot F = (\partial_x, \partial_y, \partial_z) \cdot (xy^2, yz, zx^2) = x^2 + y^2 + z. \]

Applying the divergence theorem we get
\[
\iiint_D \nabla \cdot F \, dV = \iint_S F \cdot n \, dS = \iiint_D x^2 + y^2 + z \, dV
\]
\[ = \frac{4}{3} \pi \cdot 1 = \frac{4}{3} \pi \cdot 1 = \frac{4}{3} \pi \cdot 1 = \frac{4}{3} \pi. \]

6. Calculate the flux of \( \vec{F} = \left( z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2 \right) \) across the surface \( S \), where \( S \) is the surface \( z = \sqrt{1 - x^2 - y^2} \) with normal pointing upwards. [Hint: Note that \( S \) is not a closed surface. Close \( S \) in the obvious way and then use the divergence theorem.]

**Solution**

Given the complexity of the vector field \( \vec{F} \), trying to evaluate flux \( \iint_S \vec{F} \cdot n \, dS \) directly is rather difficult and since the surface \( S \) over which the integral is to be evaluated is not closed, the divergence theorem does not apply. However, consider the closed surface \( S_{\text{tot}} \) obtained by adding a flat bottom to the hemisphere \( S \). That is, let \( S_1 \) be the surface consisting of the portion of the \( xy \)-plane within the circle \( x^2 + y^2 = 1 \), oriented with a downward pointing normal. Then let \( S_{\text{tot}} = S \cup S_1 \). Now, the integral we want to evaluate can be expressed as:

\[
\iint_S \vec{F} \cdot n \, dS = \iint_{S_{\text{tot}}} \vec{F} \cdot n \, dS - \iint_{S_1} \vec{F} \cdot n \, dS.
\]

The first integral on the right hand side is a surface integral over the closed surface \( S_{\text{tot}} \) and so the divergence theorem may be applied to this integral. Evaluating the divergence of \( \vec{F} \) yields
\[ \nabla \cdot F = (\partial_x, \partial_y, \partial_z) \cdot \left( z^2 x, \frac{y^3}{3} + \tan z, x^2 z + y^2 \right) = x^2 + y^2 + z^2. \]

Applying the divergence theorem we get
\[
\iint_{S_{\text{tot}}} \vec{F} \cdot n \, dS = \iiint_D \nabla \cdot F \, dV = \iiint_D (x^2 + y^2 + z^2) \, dV
\]
\[ = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^4 \sin \varphi \, d\rho \, d\varphi \, d\theta = \frac{2\pi}{5}. \]

To evaluate the second integral, we parameterize \( S_1 \) as follows.

\[ S_1 : \begin{cases} x = r \cos \theta & 0 \leq r \leq 1 \\ y = r \sin \theta & 0 \leq \theta < 2\pi \end{cases} \]

which leads to

\[ \vec{R} = \langle r \cos \theta, r \sin \theta, 0 \rangle \]
\[ \vec{R}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle \]
\[ \vec{R}_r = \langle \cos \theta, \sin \theta, 0 \rangle \]
\[ \vec{N} = \langle 0, 0, -1 \rangle \]

On \( S_1 \) we have \( \vec{F} \cdot \vec{N} = -r(x^2z + y^2) = -r^3 \sin^2 \theta \). Evaluating the surface integral yields
\[
\iint_{S_1} \vec{F} \cdot n \, dS = -\int_0^{2\pi} \int_0^1 r^3 \sin^2 \theta \, dr \, d\theta = -\frac{1}{4} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\theta) \, d\theta = -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}. \]
The final result is
\[
\int \int_S \vec{F} \cdot \vec{n} \, dS = \int \int_{S_{tot}} \vec{F} \cdot \vec{n} \, dS - \int \int_{S_1} \vec{F} \cdot \vec{n} \, dS = \frac{2\pi}{5} - \left( -\frac{\pi}{4} \right) = \frac{13\pi}{20}.
\]

7. Evaluate \(\int \int_S \nabla \times \vec{F} \cdot \vec{n} \, dS\), where \(\vec{F} = \langle xyz, x, e^{xy} \cos z \rangle\) and \(S\) is the hemisphere \(z = \sqrt{1 - x^2 - y^2}\) with upward pointing normal.

**Solution**
We use Stokes’s theorem to convert this surface integral to a line integral around the boundary of the surface \(S\). The boundary \(C\) of the surface \(S\) is the circle \(x^2 + y^2 = 1\) in the \(xy\)-plane. Parameterizing this circle in the obvious way, we get
\[
C : \begin{cases} 
    x = \cos \theta, & 0 \leq \theta \leq 2\pi \\
    y = \sin \theta,
\end{cases}
\]
which leads to
\[
\vec{r} = (\cos \theta, \sin \theta, 0),
\]
\[
d\vec{r} = (-\sin \theta, \cos \theta, 0) \, d\theta.
\]

Applying Stokes’ theorem we get
\[
\int \int_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) \bigg|_0^{2\pi} = \pi.
\]

8. Evaluate \(\oint_C \vec{F} \cdot d\vec{r}\), where \(\vec{F} = \langle 2xy + y + e^{x^2}, x^2 + xy - 3y + \sin(e^y), 2xz + \sinh(z^2) \rangle\) and \(C\) is the curve formed by the intersection of the cone \(z = \sqrt{x^2 + y^2}\) and the cylinder \(x^2 + y^2 = 16\) oriented in the clockwise direction when viewed from the origin.

**Solution**
One can try to evaluate this line integral directly, but given the complexity of the vector field \(\vec{F}\), it may be easier to evaluate by using Stoke’s theorem. Calculating the curl of \(\vec{F}\), we get
\[
\nabla \times \vec{F} = \begin{vmatrix}
    i & j & k \\
    \partial_x & \partial_y & \partial_z \\
    2xy + y + e^{x^2} & x^2 + xy - 3y + \sin(e^y) & 2xz + \sinh(z^2)
\end{vmatrix} = \langle 0, -2z, y - 1 \rangle.
\]

The cylinder intersects the cone in a circle of radius 4 in the plane \(z = 4\). We may take the surface \(S\) for use in Stoke’s theorem to be the inside of that circle. For \(S\) we have
\[
S : \begin{cases} 
    x = r \cos \theta, & 0 \leq r \leq 4 \\
    y = r \sin \theta, & 0 \leq \theta < 2\pi,
\end{cases}
\]
which leads to
\[
\vec{R} = \langle r \cos \theta, r \sin \theta, 4 \rangle,
\]
\[
\vec{R}_r = \langle \cos \theta, \sin \theta, 0 \rangle,
\]
\[
\vec{R}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle,
\]
\[
\vec{N} = \langle 0, 0, r \rangle.
\]
Applying Stoke’s theorem we get

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \nabla \times \mathbf{F} \cdot \mathbf{N} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^4 r(r \sin \theta - 1) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r^3}{3} \sin \theta - \frac{r^2}{2} \right]_{r=0}^{4} \, d\theta \]

\[ = \int_0^{2\pi} \left( \frac{64}{3} \sin \theta - 4 \right) \, d\theta = \left( -\frac{64}{3} \cos \theta - 8 \theta \right) \bigg|_0^{2\pi} = -16\pi \quad \blacksquare \]

9. Evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = (2xy^3, 3x^2y^2, x + 2z) \) and \( C \) is the curve consisting of the line segments joining \( A(2, 0, 0) \) to \( B(0, 1, 0) \) to \( D(0, 0, 1) \) back to \( A \).

Solution

One can try to evaluate this line integral directly, but this would require us to parameterize three separate line segments and splitting the integral into three. It may be easier to use Stoke’s theorem.

Calculating the curl of \( \mathbf{F} \), we get

\[ \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 & 3x^2y^2 & x + 2z \end{vmatrix} = (0, -1, 0). \]

The three line segments form a triangle. We may take the surface \( S \) for use in Stoke’s theorem to be the plane region inside of that triangle. A normal for the plane \( S \) is given by

\[ \mathbf{n} = \mathbf{BD} \times \mathbf{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = (1, 2, 2) \quad \Rightarrow \quad \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} = \frac{1}{3} (1, 2, 2). \]

Applying Stoke’s theorem we get

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \left( -\frac{2}{3} \right) \, dS = -\frac{2}{3} (\text{area of } S). \]

But the area of the triangle \( S \) is half of the area of the parallelogram fromed by the vectors \( \mathbf{BD} \) and \( \mathbf{BA} \). Therfore the area is

\[ \text{area of } S = \frac{1}{2} |\mathbf{BD} \times \mathbf{BA}| = \frac{1}{2} |(1, 2, 2)| = \frac{3}{2}. \]

We finally get the result

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = -\frac{2}{3} \left( \frac{3}{2} \right) = -1. \quad \blacksquare \]

10. Evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = (x + y)^2, (x + y)^2, yz^3 \) and \( C \) is the curve formed by the intersection of the cone \( z = \sqrt{x^2 + y^2} \) and the cylinder \( (x - 1)^2 + y^2 = 1 \) oriented in the counterclockwise direction when viewed from high above the \( xy \)-plane.
11. Evaluate the line integral directly, but given the complexity of the vector field $\vec{F}$, it may be easier to evaluate by using Stoke’s theorem. Calculating the curl of $\vec{F}$, we get

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ (x+y)^2 & (x+y)^2 & yz^3 \end{vmatrix} = (z^3, 0, 0).$$

The cylinder intersects the cone in a closed curve that resembles a tilted ellipse. Given the geometry of the situation, cylindrical coordinates seem most appropriate. The equation for the cylinder is $r = 2 \cos \theta$, and the equation of the cone is $z = r$. We may take for the surface $S$ is Stoke’s theorem the portion of the cone that lies within the cylinder. For $S$ we have

$$S: \begin{cases} x = r \cos \theta, & 0 \leq r \leq 2 \cos \theta \\ y = r \sin \theta, & -\frac{\pi}{2} \leq \theta < \frac{2 \pi}{2} \\ z = r \end{cases},$$

which leads to

$$\vec{n} = \langle r \cos \theta, \sin \theta, 0 \rangle,$$

$$\vec{N} = \langle -r \sin \theta, \cos \theta, 0 \rangle,$$

$$\vec{R} = \langle \cos \theta, \sin \theta, 1 \rangle.$$

Applying Stoke’s theorem we get

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \iint_D \nabla \times \vec{F} \cdot \vec{N} \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} \left( -r^4 \cos \theta \right) \, dr \, d\theta = -\frac{9}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6 \theta \, d\theta$$

$$= -\frac{9}{5} \int_0^{\frac{\pi}{2}} \left[ 1 + 3 \cos 2\theta + 3 \cos^2 2\theta + \cos^3 2\theta \right] \, d\theta$$

$$= -\frac{9}{5} \int_0^{\frac{\pi}{2}} \left[ \frac{5}{2} + 4 \cos 2\theta + \frac{3}{2} \cos 4\theta \right] \, d\theta$$

$$= -\frac{9}{5} \left[ \frac{5}{2} \theta + 2 \sin 2\theta + \frac{3}{8} \sin 4\theta - \sin^2 2\theta \right] \bigg|_0^{\frac{\pi}{2}} = -\frac{9}{5} \left[ \frac{5}{2} \left( \frac{\pi}{2} \right) \right] = -2\pi.$$

11. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle e^{x^2}, x + \sin(y^2), z \rangle$ and $C$ is the curve formed by the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the cylinder $x^2 + (y - 1)^2 = 1$ oriented in the counterclockwise direction when viewed from high above the $xy$-plane.

**Solution**

One can try to evaluate this line integral directly, but given the complexity of the vector field $\vec{F}$, it may be easier to evaluate by using Stoke’s theorem. Calculating the curl of $\vec{F}$, we get

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ e^{x^2} & x + \sin(y^2) & z \end{vmatrix} = (0, 0, 1).$$
The cylinder intersects the cone in a closed curve that resembles a tilted ellipse. Given the geometry of the situation, cylindrical coordinates seem most appropriate. The equation for the cylinder is \( r = 2 \sin \theta \), and the equation of the cone is \( z = r \). We may take for the surface \( S \) in Stoke’s theorem the portion of the cone that lies within the cylinder. For \( S \) we have

\[
S : \begin{cases} 
  x = r \cos \theta, & 0 \leq r \leq 2 \sin \theta \\
  y = r \sin \theta, & 0 \leq \theta \leq \pi \\
  z = r 
\end{cases}
\]

which leads to

\[
\begin{align*}
  R_r &= \langle \cos \theta, \sin \theta, 1 \rangle \\
  R_\theta &= \langle -r \sin \theta, r \cos \theta, 0 \rangle \\
  N &= \langle -r \cos \theta, -r \sin \theta, r \rangle
\end{align*}
\]

Applying Stoke’s theorem we get

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \iint_D \nabla \times \vec{F} \cdot \vec{N} \, d\theta d\phi = \int_0^{\pi/2} \int_0^{2\sin \theta} r \, dr \, d\theta
\]

\[
= 2 \int_0^{\pi/2} \sin^2 \theta \, d\theta = \int_0^{\pi/2} (1 - \cos 2\theta) \, d\theta = \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \pi.
\]

12. Calculate the work done by a force field \( \vec{F} = \langle x^2 + z^2, y^2 + z^2, y^2 + z^2 \rangle \) when a particle moves under its influence around the edge of the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies in the first octant, in a counterclockwise direction when viewed from above.

**Solution**

The work done by the force field \( \vec{F} \) is given by \( \oint_C \vec{F} \cdot d\vec{r} \), where \( C \) is the boundary of the portion of the sphere in the first octant. One can try to evaluate this line integral directly, but given the complexity of the vector field \( \vec{F} \) and the curve \( C \), it may be easier to evaluate by using Stoke’s theorem. Calculating the curl of \( \vec{F} \), we get

\[
\nabla \times \vec{F} = \begin{vmatrix}
  \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\
  \partial_x & \partial_y & \partial_z \\
  x^2 + z^2 & y^2 + z^2 & y^2 + z^2
\end{vmatrix} = 2 \langle y, z, 0 \rangle.
\]

The surface \( S \) is the portion of the sphere in the first octant, which we parameterize as follows:

\[
S : \begin{cases} 
  x = 2 \sin \varphi \cos \theta, & 0 \leq \varphi \leq \pi/2 \\
  y = 2 \sin \varphi \sin \theta, & 0 \leq \theta \leq \pi/2 \\
  z = 2 \cos \varphi,
\end{cases}
\]

which leads to

\[
\begin{align*}
  R_r &= \langle 2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 \cos \varphi \rangle \\
  R_\varphi &= \langle 2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi \rangle \\
  R_\theta &= \langle -2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, 0 \rangle \\
  N &= 4 \langle \sin^2 \varphi \cos \theta, \sin^2 \varphi \sin \theta, \sin \varphi \cos \varphi \rangle
\end{align*}
\]

Applying Stoke’s theorem we get

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, dS = \iint_D \nabla \times \vec{F} \cdot \vec{N} \, d\varphi d\theta
\]

\[
= 16 \int_0^{\pi/2} \int_0^{\pi/2} \sin^2 \varphi \sin \theta \cos \theta + \cos \varphi \sin \varphi d\varphi d\theta
\]

\[
= 16 \int_0^{\pi/2} \sin^2 \varphi \left( \frac{\sin^2 \theta}{2} - \cos^2 \varphi \cos \theta \right) d\varphi = 16 \int_0^{\pi/2} \sin^2 \varphi \left( \frac{1}{2} \sin \varphi + \cos \varphi \right) d\varphi
\]

\[
= 16 \int_0^{\pi/2} \left( \frac{1}{2} (1 - \cos^2 \varphi) \sin \varphi + \sin^2 \varphi \cos \varphi \right) d\varphi = \frac{32}{3}.
\]