Disintegration of positive isometric group representations on L^p-spaces

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Positivity IX
Edmonton
17 July 2017

Joint work with Jan Rozendaal
Positivity **21** (2017), 673–710

Overview

- Introduction: representations of groups and disintegration into indecomposable representations
- L^p-context: ergodic decomposition and relation with order indecomposable representations
- Direct integrals of Banach lattices
- Disintegration of group actions on L^p-spaces: spatial case
- Disintegration of group actions on L^p-spaces: general case

Disclaimer: not the final answer for all positive isometric actions

- Measures will be finite
- Group actions will leave the constants fixed
- $1 \leq p < \infty$

Unitary group representation of a locally compact group G

- Is a homomorphism $\rho : G \mapsto U(H)$ into the unitary group of a Hilbert space
- Tacitly always assumed that $g \mapsto \rho(g)x$ is continuous for all $x \in H$ (representation is strongly continuous)

Building new representations from given ones

- If ρ_1, ρ_2 are two unitary representations on H_1 and H_2, then $\rho_1 \oplus \rho_2 : G \to U(H_1 \oplus H_2)$ is another one, and H_1 and H_2 are closed invariant subspaces
- Elementary building blocks: a unitary representation on H is *indecomposable* (equivalently: irreducible) if every decomposition $H = H_1 \oplus H_2$ into closed invariant subspaces is trivial, i.e. if $H_1 = \{0\}$ or $H_1 = H$

Disintegration of $\rho : G \to U(H)$ as a direct integral

If G and H are separable, there exist a space X, a measure μ on X, families of Hilbert spaces $(H_x)_{x \in X}$ and unitary representations $(\rho_x)_{x \in X}$ of G on H_x such that

- (ρ, H) and $(\int_X^{\oplus} \rho_x \,\mathrm{d}\mu(x), \int_X^{\oplus} H_x \,\mathrm{d}\mu(x))$ are unitarily equivalent
- μ-almost ρ_x are indecomposable

Idea

- $\int_X^{\oplus} H_x \,\mathrm{d}\mu(x))$ consists of 'all' maps $s : X \to \bigsqcup_{x \in X} H_x$ such that $s(x) \in H_x$ for all x and such that

$$\int_X \|s(x)\|_{H_x}^2 \,\mathrm{d}\mu(x) < \infty$$

- $\int_X^{\oplus} \rho_x \,\mathrm{d}\mu(x)$ acts pointwise ('in each fibre')

Direct integrals of unitary representations

- Finite Hilbert sums are direct integrals for a counting measure
- In general: cannot vary $s(x)$ freely with x as for finite direct sum:
  - Need to stay square integrable
  - Measurability conditions must be met

Unitary moral in separable case

- Every unitary representation on H is built from indecomposable ones
- Equivalently: every representation as automorphisms of H is built from indecomposable ones
- Glueing formalism is that of a direct integral
- The H_x are (can be taken to be) a subspace of ℓ^2

Question

- What about representations of G as automorphisms of a Banach lattice E?
- Equivalently: what about homomorphisms $\rho : G \to B(E)$ such that every $\rho(g)$ is an isometric lattice automorphism of E?
- Still equivalently: what about *representations of* G *as positive isometries of* E?
- Can they be disintegrated into 'indecomposable' ones?

Expectation management

- Unitary theory works well for representations *in one space*: ℓ^2
- Great variety of Banach lattices
- Can't expect to cover everything; need to find 'the right class'

Indecomposability for positive representation on Banach lattice E

- If $E = E_1 \oplus E_2$ is an order direct sum of two Banach sublattices, then E_1 and E_2 are, in fact, projection bands and each other's disjoint complement
- So: a positive representation of G on E is *order indecomposable* if $\{0\}$ and E are the only invariant *projection bands*

Testing ground for disintegration issues: L^p-spaces

- Ubiquity of examples
- Good description of projection bands

Large class of examples

Whenever G acts on a space X that carries an invariant measure μ, there is a natural action of G on $L^p(X, \mu)$, given by

$$[\rho(g)]f(x) := f(g^{-1} \cdot x)$$

Note:

- ρ is a representation of G as positive isometries of the Banach lattice $\mathrm{L}^p(X, \mu)$
- If μ is finite: G leaves the constant function $\mathbf{1} \in \mathrm{L}^p(X, \mu)$ fixed

Can show (goal of the lecture)

If G is a locally compact Polish group, if $1 \leq p < \infty$, if μ is a probability measure on a set X such that $\mathrm{L}^p(X, \mu)$ is separable, and if $\rho : G \to B(\mathrm{L}^p(X, \mu))$ is a strongly continuous representation as positive isometries leaving the constants fixed, then ρ can be disintegrated into order indecomposable positive isometric representations of G on Banach lattices.

Remarks

- Disintegration uses L^p-direct integral of Banach spaces
- G does not necessarily act on X
- Need to cover that case first, though
- Polish: (homeomorphic to) separable complete metric space
- All second countable locally compact Hausdorff spaces (hence all Lie groups) are Polish

Key observation: link with ergodic decomposition

- Suppose the abstract group G acts as measurable transformation on (X, μ) with μ finite
- Projection bands: all $f \in \mathrm{L}^p(X, \mu)$ vanishing a.e. on a given measurable subset
- Invariant projection bands: all $f \in \mathrm{L}^p(X, \mu)$ vanishing a.e. on a given essentially invariant measurable subset
- So: natural representation ρ of G on $\mathrm{L}^p(X, \mu)$ is order indecomposable $\Leftrightarrow$ only trivial invariant projection bands $\Leftrightarrow$ only trivial essentially invariant measurable subsets $\Leftrightarrow$ μ is ergodic

Hope

Ergodic decomposition of μ will 'somehow' give decomposition of ρ into order indecomposable representations

Context for the moment

- Abstract G acts on X with invariant probability measure μ

First attempt

- Take an ergodic measure λ
- For $f \in \mathrm{L}^p(X, \mu)$, consider f as an element of $\mathrm{L}^p(X, \lambda)$
- Glue all these new elements together as λ ranges over de ergodic measures

Problems with first attempt

- Not clear how to glue (but see later)
- f need not be in $\mathrm{L}^p(X, \lambda)$
- Map need not even be well-defined

Toy example

- The unit circle $\mathbb{T}$ acts on the closed unit disk $\mathbb{D}$ via rotations
- Invariant probability measure μ: normalised Lebesgue measure
- Ergodic measures: rotation invariant probability measures λ_r on the orbits $r\mathbb{T}$ (circles with radius $r \in [0,1]$), viewed as measures on $\mathbb{D}$

Problems with first attempt (here: 'restriction to circles')

- If $f \in \mathrm{L}^p(\mathbb{D},\mu)$, and $r \in [0,1]$, then f need not be in $\mathrm{L}^p(\mathbb{D},\lambda_r)$: easy examples with function vanishing off $r\mathbb{T}$
- If f and g represent the same element of $\mathrm{L}^p(\mathbb{D},\mu)$, and $r \in [0,1]$, then their 'interpretations' in $\mathrm{L}^p(\mathbb{D},\lambda_r)$ need not coincide: same type of example

Outline of solution: measure on the ergodic measures and precision

- Introduce a probability measure ν *on the set of ergodic measures* $\mathcal{E}$
- If $f \in \mathcal{L}^p(X, \mu)$, i.e. if f is p-integrable with respect μ, then $f \in \mathcal{L}^p(X, \lambda)$ *for* ν*-almost all* λ *in* $\mathcal{E}$
- If $[f]_\mu = [g]_\mu$ in $\mathrm{L}^p(X, \mu)$, and if we define $[f]_\lambda = [0]_\lambda$ for the exceptional $\lambda \in \mathcal{E}$ (and naturally otherwise), and likewise for g, then $[f]_\lambda = [g]_\lambda$ *for* ν*-almost* $\lambda \in \mathcal{E}$
- So: get a map from $\mathrm{L}^p(X, \mu)$ to ν*-almost everywhere equivalence classes of sections*

$$S : \mathrm{L}^p(X, \mu) \to \bigsqcup_{\lambda \in \mathcal{E}} \mathrm{L}^p(X, \lambda)$$

For the time being

- G is a locally compact Polish group
- X is a Polish space
- G acts on X (simultaneous continuity in both variables)
- μ is an invariant Borel probability measure on X
- $\mathcal{E}$ is the (non-empty) set of ergodic Borel probability measures on X
- $\mathcal{E}$ is supplied with the induced weak*-topology from $\mathrm{C_b}(X)^*$

Be careful

- Need to keep distinction between f, $[f]_\mu$, and $[f]_\lambda$ for $\lambda \in \mathcal{E}$
- Measures are not necessarily complete
- Measurability is an issue to keep track of

Theorem (Farrell, Varadarajan)

In our Polish context (G, X), there exists a Borel measurable map $\beta : X \to \mathcal{E}$, $x \mapsto \beta_x$, such that, for all invariant μ:

1. $\beta_{gx} = \beta_x$ *for all* $x \in X$ *and* $g \in G$*;*
2. $\lambda(\beta^{-1}(\{\lambda\})) = 1$ *for all* $\lambda \in \mathcal{E}$*;*
3. *For all Borel subsets Y of X,*

$$\mu(Y) = \int_X \beta_x(Y)\,\mathrm{d}\mu(x).$$

Remarks

- Parts 1 and 2: ‘ergodic measures live on union of orbits’
- If G is compact: the ergodic measures are the push-forwards of the Haar measure to the orbits, using any point on them
- G compact, X locally compact: Seda and Wickstead (1976)

Prepare to interpret the equation: introduce measure ν on $\mathcal{E}$

- Recall: for all Borel subsets Y of X,

$$\mu(Y) = \int_X \beta_x(Y)\,\mathrm{d}\mu(x)$$

- A bit nondescriptive
- Becomes more transparent (and general!) by pushing μ forward to $\mathcal{E}$ via β:

$$\nu(A) := \mu(\beta^{-1}(A))$$

  for Borel subsets A of $\mathcal{E}$
- Now we have a Borel probability measure ν on $\mathcal{E}$
- ν does not depend on the choice for β

Theorem (—, Rozendaal (?))

In our Polish context (G, X), *with invariant* μ *on* X *and push-forward* ν *of* μ *via* β *to* $\mathcal{E}$, *we have the following:*

1. *If* $f : X \to [0, \infty]$ *is Borel measurable, then the extended function* $\lambda \mapsto \int_X f(x)\,\mathrm{d}\lambda(x)$, *with values in* $[0, \infty]$, *is Borel measurable on* $\mathcal{E}$. *In* $[0, \infty]$, *we have*

$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_{\mathcal{E}} \left(\int_X f(x)\,\mathrm{d}\lambda(x) \right) \mathrm{d}\nu(\lambda).$$

2. *If* $f \in \mathcal{L}^1(X, \mu)$, *then the set of* $\lambda \in \mathcal{E}$ *such that* $f \notin \mathcal{L}^1(X, \lambda)$ *is a Borel subset of* $\mathcal{E}$ *that has* ν*-measure zero. For* $\lambda \in \mathcal{E}$, *let* $I_f(\lambda) := \int_X f(x)\,\mathrm{d}\lambda(x)$ *if* $f \in \mathcal{L}^1(X, \lambda)$, *and let* $I_f(\lambda) := 0$ *if* $f \notin \mathcal{L}^1(X, \lambda)$. *Then* $I_f \in \mathcal{L}^1(\mathcal{E}, \nu)$, *and*

$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_{\mathcal{E}} I_f(\lambda)\,\mathrm{d}\nu(\lambda).$$

Corollary

Let $1 \leq p < \infty$, and let $f \in \mathcal{L}^p(X,\mu)$. Then the set of $\lambda \in \mathcal{E}$ such that $f \notin \mathcal{L}^p(X,\lambda)$ is a Borel subset of $\mathcal{E}$ that has ν-measure zero. For $\lambda \in \mathcal{E}$, let $s_f(\lambda) := [f]_\lambda$ if $f \in \mathcal{L}^p(X,\lambda)$, and let $s_f(\lambda) := [0]_\lambda$ otherwise. Then

$$\|[f]_\mu\|_{\mathrm{L}^p(X,\mu)} = \left(\int_{\mathcal{E}} \|s_f(\lambda)\|^p_{\mathrm{L}^p(X,\lambda)}\, \mathrm{d}\nu(\lambda)\right)^{1/p}.$$

Moving in the right direction

- Norm on $\mathrm{L}^p(X,\mu)$ has been disintegrated into the norms on the $\mathrm{L}^p(X,\lambda)$ as λ ranges over $\mathcal{E}$
- We see: *if $[f]_\mu = 0$ μ-almost everywhere, then $[f]_\lambda = [0]_\lambda$ for ν-almost all $\lambda \in \mathcal{E}$*
- Equivalence classes of sections just around the corner

So far

- Have a bundle of Banach spaces $\bigsqcup_{\lambda\in\mathcal{E}} \mathrm{L}^p(X,\lambda)$ over $\mathcal{E}$
- For each $f \in \mathcal{L}^p(X,\mu)$, we have a section

  $$s_f : \mathcal{E} \to \bigsqcup_{\lambda\in\mathcal{E}} \mathrm{L}^p(X,\lambda)$$

  that is (essentially) given by

  $$s_f = [f]_\lambda \quad (\lambda \in \mathcal{E})$$

- G acts fibrewise on sections of $\bigsqcup_{\lambda\in\mathcal{E}} \mathrm{L}^p(X,\lambda)$, *and the action in each fibre by positive isometries is order indecomposable*
- The map $f \mapsto s_f$ is G-equivariant: 'restricting to an orbit is G-equivariant'

So far—but how to continue?

- Identify sections of $\bigsqcup_{\lambda \in \mathcal{E}} \mathrm{L}^p(X, \lambda)$ that are ν-almost everywhere equal
- Yields an abstract vector space $\mathcal{S}_\nu$
- Have a (well-defined!) G-equivariant map

  $$S : \mathrm{L}^p(X, \mu) \to \mathcal{S}_\nu$$

  given by

  $$S([f]_\mu) = [s_f]_\nu$$

- Still to be done: show that $S(\mathrm{L}^p(X, \mu)) \subset \mathcal{S}_\nu$ is a Banach space in a natural way, obtained by glueing together the spaces $\mathrm{L}^p(X, \lambda)$ for $\lambda \in \mathcal{E}$

Slight extension of direct integral formalism in 'Randomly normed spaces'

- Formalism glues together Banach spaces/lattices *that need not be equal*
- But they are still connected: they contain (the image of) a common 'core' that is dense in each of them
- For example: the image of the simple functions on X is dense in $\mathrm{L}^p(X,\lambda)$ for all $\lambda \in \mathcal{E}$

Special cases (take identical spaces)

- Direct integrals of separable Hilbert spaces
- Bochner L^p-spaces

Ingredients

- A measure space $(\mathcal{E}, \nu)$ (no conditions on ν)
- A vector lattice V (the common 'core'): *think of the simple functions on X*
- A collection $\{\|\cdot\|_\lambda\}_{\lambda\in\mathcal{E}}$ of lattice seminorms on V such that $\lambda \mapsto \|x\|_\lambda$ is a measurable function on $\mathcal{E}$ for all $x \in V$: *think of the p-seminorm on the simple functions for ergodic λ*

The spaces to be glued together

- For each $\lambda \in \mathcal{E}$, let B_λ be the Banach lattice that is the completion of $V/\ker\|\cdot\|_\lambda$ in the norm induced by the seminorm $\|\cdot\|_\lambda$: *think of $\mathrm{L}^p(X,\lambda)$ for ergodic λ*
- The B_λ are glued together via the image of V: *this is what we want*

Sections

- A section is a map $s : \mathcal{E} \to \bigsqcup_{\lambda \in \mathcal{E}} B_\lambda$ such that $s(\lambda) \in B_\lambda$ for all $\lambda \in \mathcal{E}$
- A simple section is a section of the form

$$s(\lambda) = \left[\sum_{k=1}^{n} \mathbf{1}_{A_k}(\lambda) x_k \right]_\lambda \in V/\ker \|\cdot\|_\lambda \subset B_\lambda$$

  for $x_i \in V$ and measurable $A_i \subset \mathcal{E}$
- A measurable section is a section that is the pointwise limit (in the various B_λ) of simple sections

Direct integral

- Measurable sections form vector lattice with pointwise operations
- Identify measurable sections that agree ν-almost everywhere
- Gives vector lattice again
- Denoted by $\int_{\mathcal{E}}^{\oplus} B_\lambda \,\mathrm{d}\nu(\lambda)$: the direct integral of the B_λ (with respect to ν)

Important point

- For each measurable section s, the function $\lambda \to \|s(\lambda)\|_\lambda$ on $\mathcal{E}$ is measurable
- Can use this to locate normed subspaces of $\int_{\mathcal{E}}^{\oplus} B_\lambda \,\mathrm{d}\nu(\lambda)$

L^p-direct integrals

- Consider those equivalence classes $[s]_\nu \in \int_{\mathcal{E}}^{\oplus} B_\lambda \,\mathrm{d}\nu(\lambda)$ such that
$$\|[s]_\nu\|_p := \left(\int_{\mathcal{E}} \|s(\lambda)\|_\lambda^p \,\mathrm{d}\nu(\lambda)\right)^{1/p} < \infty$$
- Does not depend on the representative
- Form a normed vector lattice
- Notation:
$$\left(\int_{\mathcal{E}}^{\oplus} B_\lambda \,\mathrm{d}\nu(\lambda)\right)_{\mathrm{L}^p}$$
- Name: L^p-direct integral of the family $(B_\lambda)_{\lambda\in\mathcal{E}}$

Proposition

The L^p-direct integral

$$\left(\int_{\mathcal{E}}^{\oplus} B_\lambda \, \mathrm{d}\nu(\lambda)\right)_{\mathrm{L}^p}$$

is a Banach lattice. The set of all ν-equivalence classes of p-integrable simple sections is a dense sublattice.

Proof of completeness

- Inspired by the usual proof for L^p-spaces. Be careful with measurability
- Haydon, Levy, and Raynaud work with complete measures in 'Randomly normed spaces'

Combine with results from ergodic decomposition

- For $f \in \mathrm{L}^p(X, \mu)$, have a section $s_f : \mathcal{E} \to \bigsqcup_{\lambda \in \mathcal{E}} \mathrm{L}^p(X, \lambda)$ that is (essentially) given by

$$s_f(\lambda) = [f]_\lambda \quad (\lambda \in \mathcal{E})$$

- Can show: s_f is a measurable section of $\bigsqcup_{\lambda \in \mathcal{E}} \mathrm{L}^p(X, \lambda)$
- Have map $S : \mathrm{L}^p(X, \mu) \to \int_{\mathcal{E}}^{\oplus} \mathrm{L}^p(X, \lambda) \, \mathrm{d}\nu(\lambda)$ given by

$$S([f]_\mu) = [s_f]_\nu$$

- Disintegration of the p-norm shows that this is well-defined, and that S is an isometric embedding of $\mathrm{L}^p(X, \mu)$ into $\left(\int_{\mathcal{E}}^{\oplus} \mathrm{L}^p(X, \lambda) \, \mathrm{d}\nu(\lambda) \right)_{\mathrm{L}^p}$
- Use properties of β to verify that the image contains the ν-equivalence classes of simple sections. These are dense, so...

Theorem (disintegration: spatial case)

Let (G, X) be a Polish topological dynamical system with locally compact G, let $1 \leq p < \infty$, and let μ be an invariant Borel probability measure on X. Let $\mathcal{E}$ be the ergodic Borel probability measures on X, carrying the weak-topology from $\mathrm{C_b}(X)$. Choose a decomposition map $\beta : X \to \mathcal{E}$, and let ν be the Borel probability measure on $\mathcal{E}$ that is the push-forward of μ via β. Consider the L^p-direct integral $\left(\int_{\mathcal{E}}^{\oplus} \mathrm{L}^p(X, \lambda)\,\mathrm{d}\mu(\lambda)\right)_{\mathrm{L}^p}$ that corresponds to the vector lattice of simple functions on X and the family of p-seminorms on it that corresponds to $\mathcal{E}$.*
Then there is a natural isometric lattice isomorphism

$$S : \mathrm{L}^p(X, \mu) \to \left(\int_{\mathcal{E}}^{\oplus} \mathrm{L}^p(X, \lambda)\,\mathrm{d}\mu(\lambda)\right)_{\mathrm{L}^p}$$

under which the natural action of G on $\mathrm{L}^p(X, \mu)$ corresponds to the order indecomposable natural action of G on the fibres.

More general context

- Borel probability space (X, μ)
- Strongly continuous representation of Polish locally compact G on $\mathrm{L}^p(X, \mu)$ as positive isometries leaving $\mathbf{1}$ fixed
- So: *no underlying action of G on underlying point set X*
- Can we still disintegrate the representation into order indecomposables?

Solution: find spatial model for the situation

- If $\mathrm{L}^p(X, \mu)$ is separable: yes
- Idea behind reduction to spatial case goes back to Varadarajan (?)
- Thanks to Markus Haase for pointing this out

Lemma

Let (X,μ) be a probability space. Suppose that $\mathrm{L}^p(X,\mu)$ is separable, and that the locally compact Polish group G acts strongly continuously on $\mathrm{L}^p(X,\mu)$ as positive isometries that leave the constants fixed.
Then there exists a separable G-invariant closed subalgebra A of $(\mathrm{L}^\infty(X,\mu), \|\cdot\|_\infty)$ that contains $\mathbf{1}_X$, is dense in $\mathrm{L}^p(X,\mu)$, and is such that the restricted representation of G on $(A, \|\cdot\|_\infty)$ is strongly continuous.

Where the new space comes from

- Gelfand-Naimark (via complexification) yields compact K and unital isometric algebra and lattice isomorphism $\Phi : (A, \|\cdot\|_\infty) \to (\mathrm{C}(K), \|\cdot\|_\infty)$
- A is separable, so K is compact metrisable space: Polish

Take a few steps to transfer everything to K (details omitted)

- G acts strongly continuously on A: *gives action of G on K that is continuous in both variables*
- Then $\Phi : A \to \mathrm{C}(K)$ is G-equivariant by construction
- Riesz representation theorem gives Borel probability measure $\widetilde{\mu}$ on K such that

$$\int_K \Phi(f)\,\mathrm{d}\widetilde{\mu} = \int_X f\,\mathrm{d}\mu \quad (f \in A)$$

- $\widetilde{\mu}$ is G-invariant since G acts as isometries on $\mathrm{L}^p(X,\mu)$
- Φ is isometric for the p-norms corresponding to $\widetilde{\mu}$ on K and to μ on X
- Hence (A is dense in $\mathrm{L}^p(X,\mu)$!) Φ extends to G-equivariant isometric lattice isomorphism between $\mathrm{L}^p(K,\widetilde{\mu})$ and $\mathrm{L}^p(X,\mu)$

Mission accomplished

- Have found an alternative model for the representation of G on $\mathrm{L}^p(X,\mu)$: the representation of G on $\mathrm{L}^p(K,\widetilde{\mu})$ *that originates from the action of G on K*
- This we can handle...

Theorem (disintegration: general case)

Let G be a locally compact Polish group, let $1 \leq p < \infty$, and let μ be a probability measure on a set X such that $\mathrm{L}^p(X,\mu)$ is separable. If $\rho : G \to B(\mathrm{L}^p(X,\mu))$ is a strongly continuous representation as positive isometries leaving the constants fixed, then ρ is isometrically lattice equivalent to an L^p-direct integral of similar representations on L^p-spaces (for Borel probability measures on a common compact metric space) that are order indecomposable.