Positivity IX, July 17-21, 2017
University of Alberta,
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Lie-Trotter product formula for (locally) equicontinuous (and tight) Markov operators

Joint work with S.Hille

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Outline

- Lie-Trotter product formula
  - A bit of history [1875-2001]
  - New result by Kuhnemund and Wacker [2001]
  - Our goal
- Our setting
  - Markov operators
  - Equicontinuity and tightness
  - Assumptions
- Main result-what is new
  - Convergence of the scheme
  - Our approach

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Lie-Trotter product formula

- A bit of history [1875-2001]

Sophius Lee

Lie product product formula

$$e^{A+B} = \lim_{n\to\infty} \left(e^{A/N} e^{B/N}\right)^N$$

1875 1959 1970 2001 2016

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Lie-Trotter product formula

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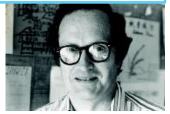
1875 1959 1970 2001 2016

$$S_{a,t} = \lim_{h\to\infty} \left(T_h T'_{ah}\right)^{t/h}$$

Hale Trotter
Lie-Trotter product formula

Strong continuity, generators



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Lie-Trotter product formula

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$$e^{A+B} = \lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n$$

Paul Chernoff

Chernoff product formula

Generalizations of Trotter product formula

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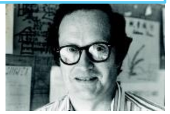

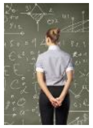

Franziska Kuhnemund
Markus Wacker

$$U(t)f = \lim_{n\to\infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^{n} f$$

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Lie-Trotter product formula

- New result by Kuhnemund and Wacker [2001]

- **1991** Engel-Nagel counterexample (...but actually 1985 Goldstein)

2001 2016

Franziska Kuhnemund
Markus Wacker


- C_0 -semigroups
- exponentially bounded: $||T(t)|| \le Me^{\omega t}$
- locally Trotter stable:

$$\left\| \left(T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right)^n \right\| \le M t_0$$

- commutator condition:

$$||T(t)S(t)f - S(t)T(t)f|| \le t^{\alpha} M |||f|||$$

$$\boldsymbol{U(t)f = \lim_{n\to\infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^n f}$$

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Lie-Trotter product formula

- Our goal (2016)

Generalization of Lie-Trotter product formula to semigroups of Markov operators on spaces of measures

2001 2016

Sander Hille
MZ

Universiteit Leiden

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Lie-Trotter product formula

- Motivation

Generalization of Lie-Trotter product formula to semigroups of Markov operators on spaces of measures

2001 2016

Sander Hille

MZ

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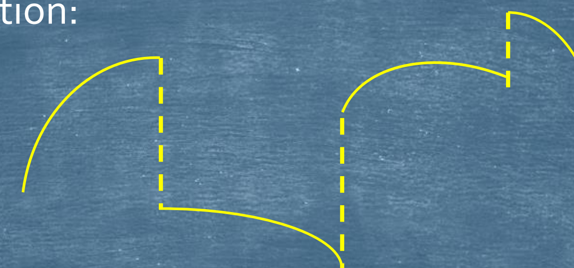
Motivation:



Piecewise deterministin Markov processes

Switching deterministic and stochastic processes

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  - Markov operators on spaces on measures
  - Equicontinuity
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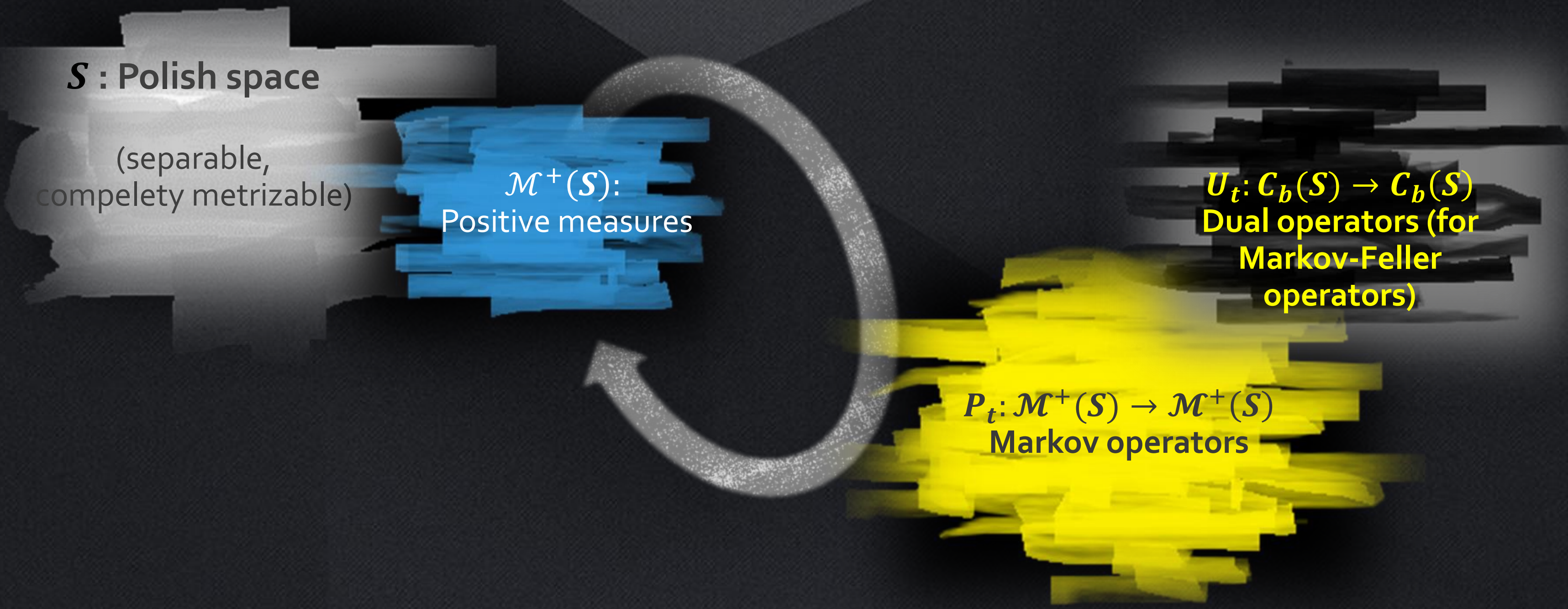
Our setting

- Markov operators on spaces of measures



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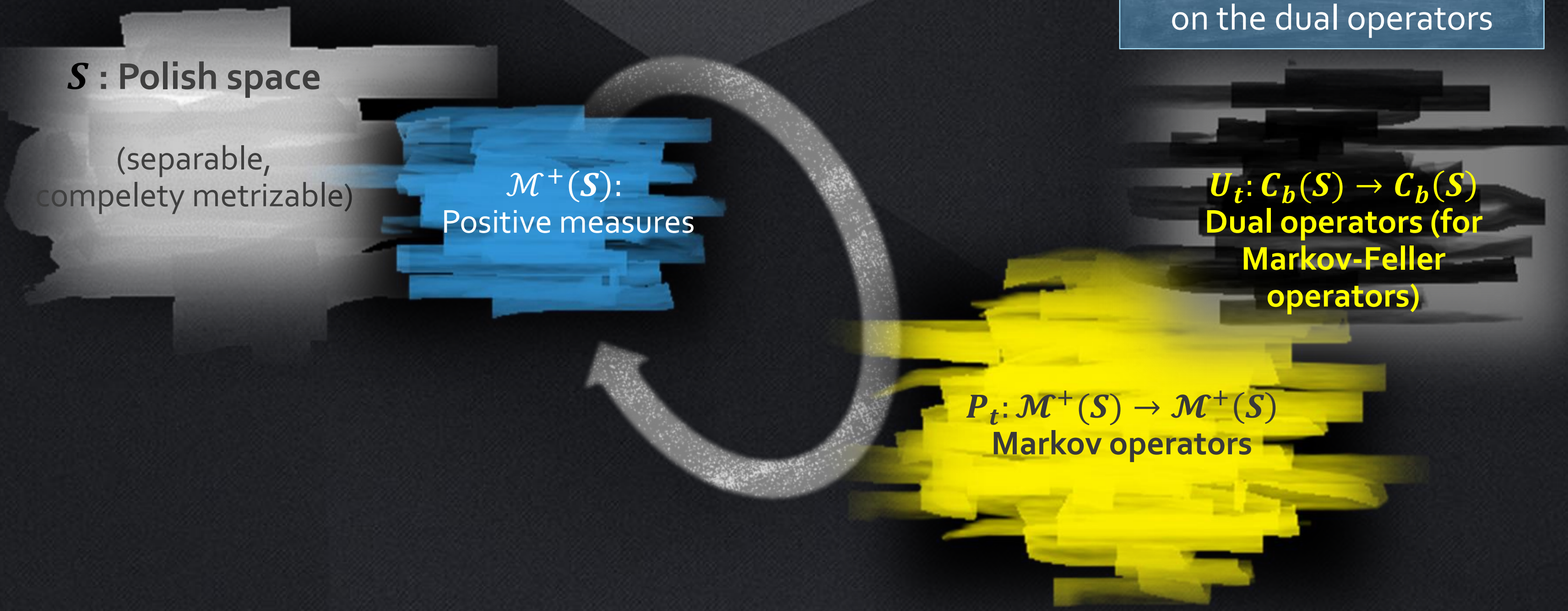
Our setting

- Markov operators on spaces of measures

Most papers concentrate on the dual operators



Our setting

- Markov operators on spaces of measures

Most papers concentrate on the dual operators

$\boldsymbol{S}$: **Polish space**

(separable, compelety metrizable)

$\mathcal{M}^+(\boldsymbol{S})$:
Positive measures

$\boldsymbol{U_t}: \boldsymbol{C_b(S)} \to \boldsymbol{C_b(S)}$
Dual operators (for Markov-Feller operators)

$\boldsymbol{P_t}: \boldsymbol{\mathcal{M}^+(S)} \to \boldsymbol{\mathcal{M}^+(S)}$
Markov operators

We concentrate on Markov **operators**

Our setting

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We concentrate on Markov **operators**

Additive, $\mathbb{R}_+$-homogenous, $||\cdot||_{TV}$-norm preserving

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Markov operators

We concentrate on Markov **operators**

Additive, $\mathbb{R}_+$-homogenous, $||P\mu||_{TV} = ||\mu||_{TV}, \mu \in \mathcal{M}^+(S)$

$||\mu||_{TV} = |\mu|(S)$

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Our setting

- Equicontinuity and tightness

Our assumptions:

- A1: (P_t^1), (P_t^2)- locally ($t \in [0, \delta]$) equicontinuous and tight
- A2: $(\mathrm{P}_{\frac{t}{n}}^1 \mathrm{P}_{\frac{t}{n}}^2)^{\mathrm{n}}$ - locally equicontinuous and tight
- A3: stability

$$\left|\left|(\mathrm{P}_{\frac{t}{n}}^1 \mathrm{P}_{\frac{t}{n}}^2)^{n} \mu_0\right|\right|_{BL,d_E}^* \leq \mathrm{C}|\mu_0|_{\mathrm{M}_0}$$

- A4: commutator condition

$$\left|\left|\mathrm{P}_t^1 \mathrm{P}_t^2 \mu_0 - \mathrm{P}_t^2 \mathrm{P}_t^1 \mu_0\right|\right|_{\mathrm{BL,d_E}}^* \leq t\omega_E(t)|\mu_0|_{\mathrm{M}_0}$$

ω_E –nondecreasing, continuous, Dini condition

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ω_E −nondecreasing, continuous, Dini condition

A family $F \in C(T,(S,d))$ is **equicontinuous at point $t \in T$** if $\forall \varepsilon > 0\ \exists U_\varepsilon$ s.t. $d_S(f(t), f(t')) < \varepsilon\ \forall t' \in U_\varepsilon\ \forall f \in F$

The family F is **equicontinuous** if and only if it is **equicontinuous at every point.**

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ω_E −nondecreasing, continuous, Dini condition

$\|f\|_{BL,d_E} = \|f\|_\infty + |f|_{\{Lip,d_E\}}$

$|f|_{\{Lip,d_E\}}$-Lipschitz constant

$\|\cdot\|_{BL,d_E}^*$-Dudley norm

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ω_E −nondecreasing, continuous, Dini condition

A family $P_t, t \geq 0$ is **tight** if for every positive measure $\mu \in \mathcal{M}^+(S)$, $\{P_t\mu: t \geq 0\}$ is **uniformly tight.**
$\{P_t\mu: t \geq 0\}$ is **uniformly tight** if $\forall \epsilon > 0 \exists K_\epsilon -$ $compact\ s.t. |P_t\mu|(S \backslash K_\epsilon) < \epsilon$ for all $t \geq 0$.

uniform tightness=**relative compactness of orbits** (in Dudley norm)

Our setting

- Comparison with K-W

Our assumptions:

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ω_E –nondecreasing, continuous, Dini condition

Reminder (K-W setting):

- C_0 -semigroups
- exponentially bounded: $\|T(t)\| \leq Me^{\omega t}$
- locally Trotter stable:

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Markov semigroups are usually **neither strongly continuous nor bounded**.

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Main theorem

Reminder:

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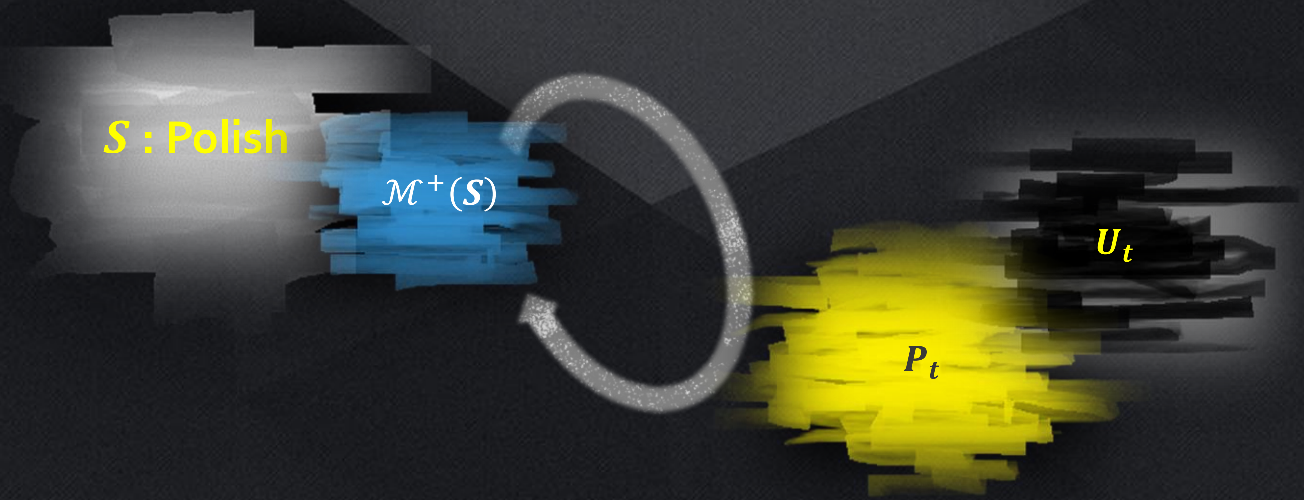
$$\left\|P_t^1 P_t^2 \mu_0 - P_t^2 P_t^1 \mu_0\right\|_{BL,d_E}^* \leq t\omega(t)\ |\mu_0|_{M_0}$$

Main result

Let $(P_t^1)_t, (P_t^2)_t$ be semigroups of regular Markov-Feller operators. Assume that A1-A4 hold. Let $\mu \in \mathcal{M}^+(S)$. Then there exists $\nu \in \mathcal{M}^+(S)$ such that

$$\left\|\left(P_{\frac{t}{n}}^1 P_{\frac{t}{n}}^2\right)^n \mu - \nu\right\|_{BL,d}^* \to 0 \ as\ n \to \infty.$$

Our approach and tools we use



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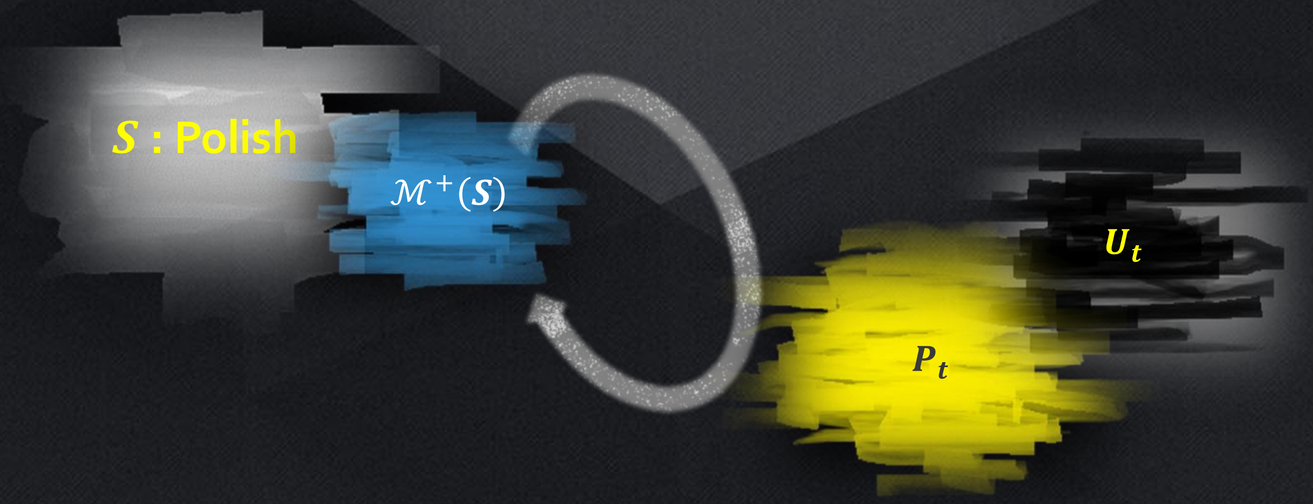

<u>We can show that:</u>

- A composition of equicontinuous Markov operators is equicontinuous

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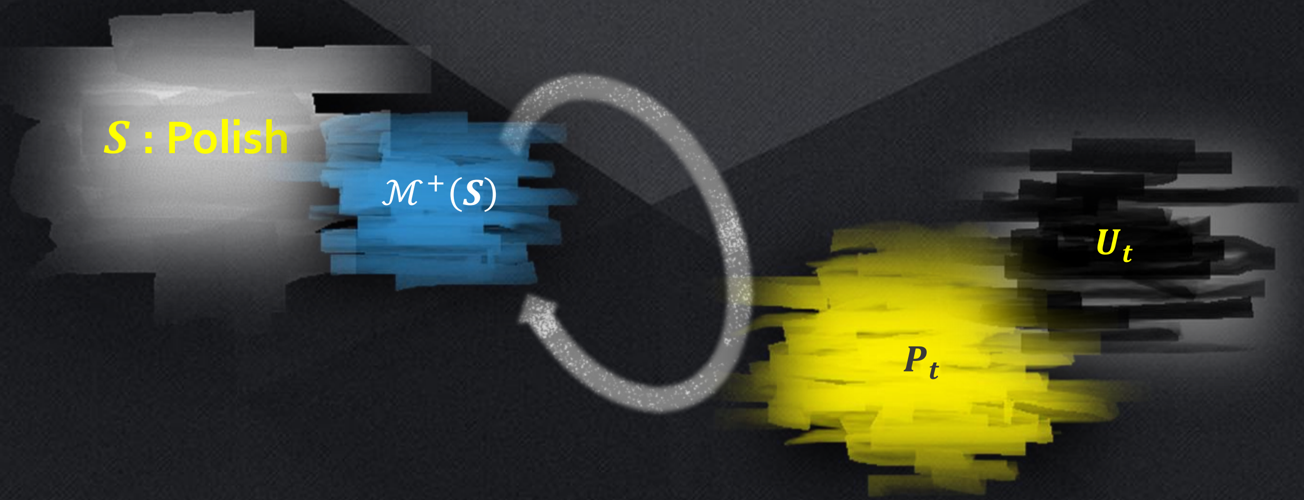

<u>Topological properties:</u>

- Equicontinuity is equal to equicontinuity on compact sets

<u>We can show that:</u>

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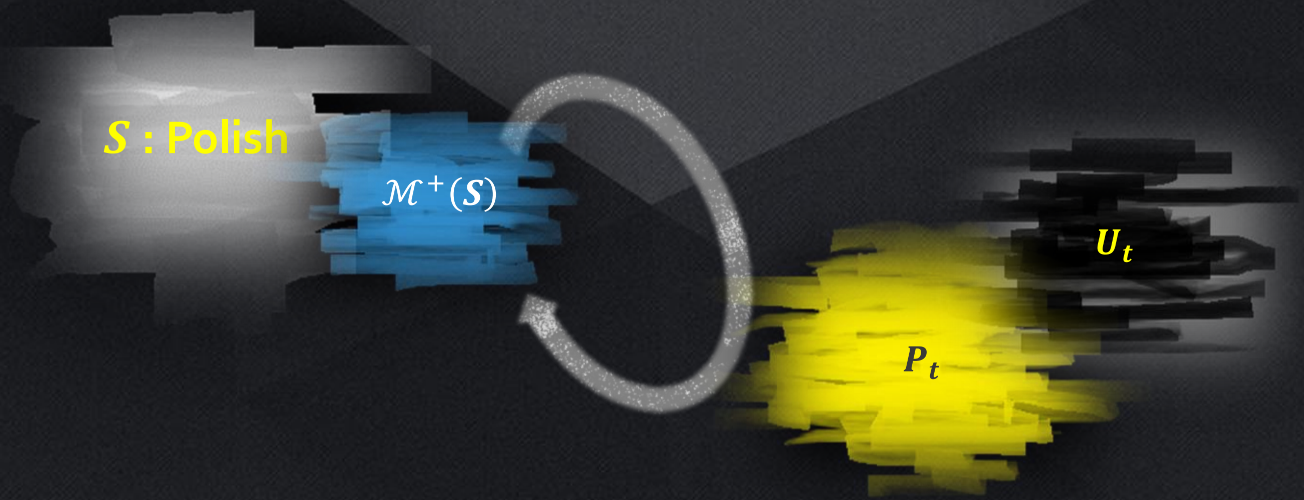
Our approach and tools we use



<u>Schur-like property:</u>

- Weak convergence of $| \ |_{TV}$ – bounded sequences in the space of **signed measures** implies strong convergence

S.Hille, T.Szarek, D.Worm, MZ „On a *Schur-like Property for Spaces of Measures*" submitted, available on Arxiv

<u>Topological properties:</u>

- Equicontinuity is equal to equicontinuity on compact sets

<u>We can show that:</u>

- A composition of equicontinuous Markov operators is equicontinuous

Conclusion

- Generalizations of Kuhnemund-Wacker and Colombo-Corli setting
- No assumptions on generators/domains of generators (motivated by an example in Engel-Nagel/Goldstein)

Open problems:

- Relations between generators/domains of generators and equicontinuity/tightness (in progress)
- ...and much more

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Thank you!

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