Lie-Trotter product formula for (locally) equicontinuous (and tight) Markov operators

Joint work with S. Hille
Outline

- Lie-Trotter product formula
  - A bit of history [1875-2001]
  - New result by Kuhnemund and Wacker [2001]
  - Our goal

- Our setting
  - Markov operators
  - Equicontinuity and tightness
  - Assumptions

- Main result-what is new
  - Convergence of the scheme
  - Our approach
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A bit of history [1875-2001]

\[e^{A+B} = \lim_{n \to \infty} (e^{A/N} e^{B/N})^N\]
A bit of history [1875-2001]
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Lie-Trotter product formula

Sophius Lie
Lie product formula

Paul Chernoff
Chernoff product formula

Generalizations of Trotter product formula

Hale Trotter
Lie-Trotter product formula

Strong continuity, generators

Sophius Lie
Lie product formula

Paul Chernoff
Chernoff product formula

Strong continuity, generators

$e^{A+B} = \lim_{n \to \infty} \left( e^{A/n} e^{B/n} \right)^n$

$S_{a,t} = \lim_{h \to \infty} (T_h T'_ah)_{t/h}$
Lie-Trotter product formula

A bit of history [1875-2001]

Sophus Lie
Lie product formula

Paul Chernoff
Chernoff product formula

Generalizations of Trotter product formula

Hale Trotter
Lie-Trotter product formula

Franziska Kuhnemund
Markus Wacker

Strong continuity, generators

\[ e^{A+B} = \lim_{n \to \infty} \left( e^{A/n} e^{B/n} \right)^n \]

\[ S_{a,t} = \lim_{h \to 0} \left( T_h T_{a,h} \right)^{t/h} \]

\[ U(t)f = \lim_{n \to \infty} \left[ f \left( \frac{t}{n} \right) S \left( \frac{t}{n} \right) \right]^n f \]
- **New result by Kuhnemund and Wacker [2001]**

- **1991 Engel-Nagel counterexample (...but actually 1985 Goldstein)**

- **$C_0$-semigroups**
  - exponentially bounded: $\|T(t)\| \leq Me^{\omega t}$
  - locally Trotter stable:
    $$\left\| \left( T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right)^n \right\| \leq Mt_0$$
  - commutator condition:
    $$\|T(t)S(t)f - S(t)T(t)f\| \leq t^\alpha M|||f|||$$

- **$U(t)f = \lim_{n \to \infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^n f$**
Generalization of Lie-Trotter product formula to semigroups of Markov operators on spaces of measures

Our goal (2016)
Lie-Trotter product formula

Motivation:

Piecewise deterministic Markov processes

Switching deterministic and stochastic processes

Generalization of Lie-Trotter product formula to semigroups of Markov operators on spaces of measures
Lie-Trotter product formula
  - A bit of history [1875-2001]
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Our setting
  - Markov operators on spaces on measures
  - Equicontinuity
  - Assumptions

Main result - what is new
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Our setting

- Markov operators on spaces of measures

$S$: Polish space (separable, completely metrizable)

$P_t: \mathcal{M}(S) \rightarrow \mathcal{M}(S)$: Markov operators

$U_t: \mathcal{C}_b(S) \rightarrow \mathcal{C}_b(S)$: Dual operators (for Markov-Feller operators)

$\mathcal{M}^+(S)$: Positive measures
Our setting

Markov operators on spaces of measures

\( \mathcal{M}^+ (S) \): Positive measures

\( U_t : \mathcal{C}_b (S) \to \mathcal{C}_b (S) \)
Dual operators (for Markov-Feller operators)

\( P_t : \mathcal{M}^+ (S) \to \mathcal{M}^+ (S) \)
Markov operators

Most papers concentrate on the dual operators
Markov operators on spaces of measures

Our setting:

- Markov operators on spaces of measures

Most papers concentrate on the dual operators

We concentrate on Markov operators

\(\mathcal{M}^+ (S)\): Positive measures

\(U_t: C_b (S) \rightarrow C_b (S)\)

Dual operators (for Markov-Feller operators)

\(P_t: \mathcal{M}^+ (S) \rightarrow \mathcal{M}^+ (S)\)

Markov operators

\(\mathcal{S}\): Polish space (separable, completely metrizable)
Markov operators on spaces of measures

Our setting

\[ S : \text{Polish space} \quad (\text{separable, \ complete \ metrizable}) \]

\[ P_t : \mathcal{M}^+(S) \to \mathcal{M}^+(S) \quad \text{Markov operators} \]

Most papers concentrate on the dual operators

\[ U_t : C_b(S) \to C_b(S) \quad \text{Dual operators (for Markov-Feller operators)} \]

We concentrate on Markov operators

Additive, \( \mathbb{R}_+ \)-homogenous, \( \| \cdot \|_{TV} \)-norm preserving
Our setting

- Markov operators on spaces of measures

\( S : \) Polish space (separable, completely metrizable)

\( \mathcal{M}^+(S) : \) Positive measures

\( P_t : \mathcal{M}^+(S) \to \mathcal{M}^+(S) \)

Markov operators

Most papers concentrate on the dual operators

\( U_t : C_b(S) \to C_b(S) \)

Dual operators (for Markov-Feller operators)

\( P_t : \mathcal{M}^+(S) \to \mathcal{M}^+(S) \)

Markov operators

Additive, \( \mathbb{R}_+ \)-homogenous

\[ ||\mu||_{TV} = |\mu|(S) \]

Additive, \( \mathbb{R}_+ \)-homogenous,

\[ ||P\mu||_{TV} = ||\mu||_{TV}, \mu \in \mathcal{M}^+(S) \]
Our setting

- Equicontinuity and tightness

**Our assumptions:**

- **A1:** \( (P_t^1), (P_t^2) \) - locally \( t \in [0, \delta] \) equicontinuous and tight

- **A2:** \( (P_t^n P_t^n)^n \) - locally equicontinuous and tight

- **A3:** stability
  \[ \left\| (P_t^n P_t^n)^n \mu_0 \right\|_{\text{BL},d_E}^* \leq C |\mu_0|_{\text{M}_0} \]

- **A4:** commutator condition
  \[ \left\| P_t^n P_t^2 \mu_0 - P_t^2 P_t^n \mu_0 \right\|_{\text{BL},d_E}^* \leq t \omega_E(t) |\mu_0|_{\text{M}_0} \]

\( \omega_E \) - nondecreasing, continuous, Dini condition
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  \]

\(\omega_E\) - nondecreasing, continuous, Dini condition

A family \(F \in C(T, (S, d))\) is **equicontinuous at point** \(t \in T\) if \(\forall \varepsilon > 0 \exists U_\varepsilon\) s.t. \(d_S(f(t), f(t')) < \varepsilon \ \forall t' \in U_\varepsilon\ \forall f \in F\)

The family \(F\) is **equicontinuous** if and only if it is **equicontinuous at every point**.
Our setting

Equicontinuity and tightness

Our assumptions:

- **A1**: $(P^1_t, P^2_t)$ - locally $(t \in [0, \delta])$ equicontinuous and tight

- **A2**: $(P^1_t P^n_2)^n$ - locally equicontinuous and tight

- **A3**: stability

\[
\left\| \left( P^n_1 P^n_2 \right)^n \mu_0 \right\|_{BL, d_E}^* \leq C |\mu_0|_{M_0}
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- **A4**: commutator condition

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\left\| P^n_1 P^n_2 \mu_0 - P^n_2 P^n_1 \mu_0 \right\|_{BL, d_E}^* \leq t \omega_E(t) |\mu_0|_{M_0}
\]

$\omega_E$ — nondecreasing, continuous, Dini condition

\[
|f|_{BL, d_E} = |f|_\infty + |f|_{\{Lip, d_E\}}
\]

$|f|_{\{Lip, d_E\}}$ - Lipschitz constant

$||\cdot||^*_{BL, d_E}$ - Dudley norm
Our setting

- Equicontinuity and tightness

**Our assumptions:**

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- **A2:** \((P_t^1 P_t^2)^n\) - locally equicontinuous and tight

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  \[
  \left\| P_t^1 P_t^2 \mu_0 - P_t^2 P_t^1 \mu_0 \right\|_{BL,d_E}^* \leq t \omega_E(t) |\mu_0|_{M_0}
  \]

\(\omega_E\) - nondecreasing, continuous, Dini condition

A family \(P_t, t \geq 0\) is **tight** if for every positive measure \(\mu \in \mathcal{M}^+(S)\), \(\{P_t \mu : t \geq 0\}\) is uniformly tight.

\(\{P_t \mu : t \geq 0\}\) is uniformly tight if \(\forall \epsilon > 0 \exists K_\epsilon - compact\ s.t.\ |P_t \mu|(S \backslash K_\epsilon) < \epsilon\) for all \(t \geq 0\).

Uniform tightness = **relative compactness of orbits** (in Dudley norm)
Comparison with K-W

**Our assumptions:**
- **A1:** \((P_t^1), (P_t^2)\) - locally \((t \in [0, \delta])\) equicontinuous and tight
- **A2:** \((\frac{P_t^1 P_t^2}{n})^n\) - locally equicontinuous and tight
- **A3:** stability
  \[
  \left\| \left( \frac{P_t^1 P_t^2}{n} \right)^n \mu_0 \right\|_{*, BL, dE}^* \leq C |\mu_0|_{M_0}
  \]
- **A4:** commutator condition
  \[
  \left\| \left( P_t^1 P_t^2 \mu_0 - P_t^2 P_t^1 \mu_0 \right) \right\|_{BL, dE}^* \leq t \omega_E(t) |\mu_0|_{M_0}
  \]
  \(\omega_E\) – nondecreasing, continuous, Dini condition

**Reminder (K-W setting):**
- \(C_0\) -semigroups
- exponentially bounded: \(|T(t)| \leq Me^{\omega t}\)
- locally Trotter stable:
  \[
  \left\| \left( T\left( \frac{t}{n} \right) S\left( \frac{t}{n} \right) \right)^n \right\| \leq Mt_0
  \]
- commutator condition:
  \[
  \|T(t)S(t)f - S(t)T(t)f\| \leq t^\alpha M ||f||
  \]
Our setting

- Comparison with K-W

**Our assumptions:**

- **A1:** $(P^1_t)$, $(P^2_t)$ - locally $(t \in [0, \delta])$ equicontinuous and tight
- **A2:** $(\frac{P^1_t P^2_t}{n})^n$ - locally equicontinuous and tight
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  \]

$\omega_E$ - nondecreasing, continuous, Dini condition

Reminder (K-W setting):

- $C_0$-semigroups
- exponentially bounded: $\|T(t)\| \leq M e^{\omega t}$
- locally Trotter stable:
  \[
  \left\| \left( \frac{T(t)}{n} S \left( \frac{t}{n} \right) \right)^n \right\| \leq M t_0
  \]
- commutator condition:
  \[
  \|T(t)S(t)f - S(t)T(t)f\| \leq t^\alpha M \|f\|$

Markov semigroups are usually neither strongly continuous nor bounded.
Lie-Trotter product formula
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Main result—what is new
- Convergence of the scheme
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Main result

Let \((P^1_t)_t, (P^2_t)_t\) be semigroups of regular Markov-Feller operators. Assume that \(A_1-A_4\) hold. Let \(\mu \in \mathcal{M}^+(S)\). Then there exists \(\nu \in \mathcal{M}^+(S)\) such that

\[
\left\| \left( \frac{P^1_t P^2_t}{n} \right)^n \mu - \nu \right\|_{BL,d}^* \to 0 \text{ as } n \to \infty.
\]

Reminder:

\(A_1\): \((P^1_t), (P^2_t)\) - locally equicontinuous and tight

\(A_2\): \((P^1_t P^2_t)^n\) - locally \(n\) equicontinuous and tight

\(A_3\): stability

\[
\left\| \left( \frac{P^1_t P^2_t}{n} \right)^n \mu_0 \right\|_{BL,d}^* \leq C |\mu_0|_{M_0}
\]

\(A_4\): commutator condition

\[
\left\| \left| \frac{P^1_t P^2_t}{n} \mu_0 - \frac{P^2_t P^1_t}{n} \mu_0 \right| \right\|_{BL,d}^* \leq t \omega(t) |\mu_0|_{M_0}
\]
Our approach and tools we use
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We can show that:

- A composition of equicontinuous Markov operators is equicontinuous
**Our approach and tools we use**

**Topological properties:**
- Equicontinuity is equal to equicontinuity on compact sets

**We can show that:**
- A composition of equicontinuous Markov operators is equicontinuous
Our approach and tools we use

**Schur-like property:**

- Weak convergence of $| |_{TV}$ bounded sequences in the space of signed measures implies strong convergence

S. Hille, T. Szarek, D. Worm, MZ „On a Schur-like Property for Spaces of Measures“ submitted, available on Arxiv

**Topological properties:**

- Equicontinuity is equal to equicontinuity on compact sets

**We can show that:**

- A composition of equicontinuous Markov operators is equicontinuous
Conclusion

- Generalizations of Kühnemund-Wacker and Colombo-Corli setting

- No assumptions on generators/domains of generators (motivated by an example in Engel-Nagel/Goldstein)

Open problems:

- Relations between generators/domains of generators and equicontinuity/tightness (in progress)

- ...and much more
Thank you!

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