Disjointness preserving C_0-semigroups and local operators on ordered Banach spaces

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Theorem (W. Arendt, 1986)

Let A be the generator of a disjointness preserving semigroup $T(t)_{t\geq 0}$ on a Banach lattice X. Then A is local (i.e. $x \perp y$ implies $Ax \perp y$, $x \in \mathcal{D}(A), y \in X$).

Theorem (W. Arendt, 1986)

Let A be the generator of C_0-semigroup $T(t)_{t\geq 0}$ on Banach lattice X with order continuous norm. TFAE:

(i) *$T(t)_{t\geq 0}$ is a semigroup of lattice homomorphism.*

(ii) *$\mathcal{D}(A)$ is a sublattice and A is local.*

Outline

1 Normed partially ordered vector spaces

2 Local operators

3 Disjointness preserving C_0-semigroups

- (X, K) partially ordered vector space (POVS), for every finite $M \subseteq X$, the set of all upper bounds of M defined by $M^{\mathrm{u}} = \{x \in X \colon x \geq m, \forall m \in M\}$, the set of all lower bounds by M^{l}.

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- X is pre-Riesz space if $\forall x, y, z \in X$, $\{x + y, x + z\}^{\mathrm{u}} \subseteq \{y, z\}^{\mathrm{u}}$ implies $x \in K$, every directed Archimedean POVS is pre-Riesz.

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- Semimonotone if $\exists M \in \mathbb{R}$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $\|x\| \leq M\|y\|$.

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- X is pre-Riesz space if $\forall x, y, z \in X$, $\{x + y, x + z\}^{\mathrm{u}} \subseteq \{y, z\}^{\mathrm{u}}$ implies $x \in K$, every directed Archimedean POVS is pre-Riesz.
- X is directed, a seminorm $\|\cdot\|$ on X is regular if $\|x\| = \inf\{\|y\| : \ -y \leq x \leq y\}$, $x \in X$.
- Semimonotone if $\exists M \in \mathbb{R}$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $\|x\| \leq M\|y\|$.
- $x, y \in X$ are called disjoint, in symbol $x \perp y$, if $\{x + y, -x - y\}^{\mathrm{u}} = \{x - y, -x + y\}^{\mathrm{u}}$.

- $D \subseteq X$ is order dense in X if $x = \inf\{d \in D \colon x \leq d\}$, $x \in X$.

Theorem (M. van Haandel, 1993)

Let X be a POVS, TFAE:

(i) *X is a pre-Riesz space.*

(ii) *There exist a vector lattice X^ρ and a bipositive linear map $i \colon X \to X^\rho$ such that $i[X]$ is order dense in X^ρ, and generates X^ρ as a vector lattice. Moreover, all spaces X^ρ are isomorphic as vector lattices.*

- (X^ρ, i) is called the Riesz completion of X.

Lemma

If one of following statements holds,

(i) (X, K) *is a pre-Riesz space with a regular norm* $\|\cdot\|$ *such that* K *is closed.*

(ii) $(X, K, \|\cdot\|)$ *is an ordered Banach space with semimonotone norm.*

Then every band in X *is closed.*

- $B \subseteq X$, $B^d = \{x \in X : x \perp y, \forall y \in B\}$.
- $B \subseteq X$ is a **band** in pre-Riesz space X if $B^{dd} = B$.

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Definition

Let X be a POVS and let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a linear operator.

(i) T is called local if for every $x \in \mathcal{D}(T)$, $y \in X$ with $x \perp y$ it follows that $Tx \perp y$.

(ii) T is called band preserving if for every band B in X one has $T(B \cap \mathcal{D}(T)) \subseteq B$.

X a pre-Riesz space, $T\colon X \supseteq \mathcal{D}(T) \to X$ a linear operator.

- Properties
  - T is local $\Leftrightarrow$ T is band preserving.
  - If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon \mathcal{D}(T) \supseteq X \to X$ are local operators and $\alpha, \beta \in \mathbb{R}$, then $\alpha S + \beta T\colon X \supseteq \mathcal{D}(S) \cap \mathcal{D}(T) \to X$ is a local operator.
  - If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon X \supseteq \mathcal{D}(T) \to \mathcal{D}(S) \subseteq X$ are local operators, then $ST\colon X \supseteq \mathcal{D}(T) \to X$ is a local operator.

- Let X, Y be pre-Riesz spaces, $i\colon X \to Y$ is a Riesz* homomorphism, if for a finite $M \subseteq X$ we have $i[M^{\mathrm{ul}}] \subseteq i[M]^{\mathrm{ul}}$.

Theorem (M. van Haandel, 1993)

Let X, Y be pre-Riesz spaces with Riesz completions (X^ρ, i_X) and (Y^ρ, i_Y) respectively. Let $T : X \to Y$ be a linear operator. Then there exists a linear lattice homomorphism $T_\rho : X^\rho \to Y^\rho$ if and only if T is a Riesz homomorphism such that $T_\rho \circ i_X = i_Y \circ T$.*

- $$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X^\rho & \xrightarrow{T_\rho} & Y^\rho \end{array}$$

Lemma

Let X, Y be pre-Riesz spaces, $i \colon X \to Y$ a bipositive Riesz homomorphism.*

Then for every $x, y \in X$ we have $x \perp y \iff i(x) \perp i(y)$.

Proposition

Let X be a pre-Riesz space, $T \colon X \supseteq \mathcal{D}(T) \to X$ a bijective linear operator, $i \colon \mathcal{D}(T) \to X$ is a Riesz homomorphism.*

If T and T^{-1} are positive, T is local, then T^{-1} is also local.

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Theorem (W. Arendt, 1986)

Let A be the generator of a disjointness preserving semigroup $T(t)_{t\geq 0}$ on a Banach lattice X. Then A is local.

Theorem

Let X be an ordered Banach space with semimonotone norm, $T(t)_{t\geq 0} \in \mathcal{L}(X)$ a disjointness preserving C_0-semigroup with generator A. Then A is local.

Example

Let A be the second derivative operator that A is local. The one-dimensional diffusion semigroup generated by A is given by

$$T(t)f(x) = \int_0^1 K_t(x,y)f(x)dy,$$

with kernel

$$K_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos(\pi n x)\cdot\cos(\pi n y).$$

$K_t(\cdot,\cdot)$ is a positive, continuous on $[0,1]^2$.

However, $T(t)_{t\geq 0}$ is not disjointness preserving on $\mathrm{C}[0,1]$.

Theorem

Let X be an ordered Banach space with a semimonotone norm. If $A \in \mathcal{L}(X)$ is local, then $\exp(tA)$ is local for every $t \in \mathbb{R}$.

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Example

(i) Translation Semigroup

$X := \mathrm{C}_{\mathrm{ub}}(\mathbb{R})$, $T(t)x(s) := x(s+t), s \in \mathbb{R}, x \in X, t \geq 0$. $T(t)$ is a C_0-semigroup with generator A given by, $Ax := x', x \in \mathcal{D}(A)$. Then A is local (and unbounded), T is disjointness preserving, but not local.

Example

(ii) Multiplication Semigroup

$X := \mathrm{C}_0(\Omega)$, Ω is a locally compact Hausdorff space, $q\colon \Omega \to \mathbb{R}$ be continuous and bounded above. Define $T_q(t)_{t\geq 0}\colon X \to X$ by

$$T_q(t)x = e^{tq(t)}x,\ \ x \in X.$$

$T_q(t)_{t\geq 0}$ is a C_0-semigroup with generator A given by $Ax = qx, x \in \mathcal{D}(A)$. Then A is local and $T_q(t)$ is local for every $t \in [0, \infty)$.

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Theorem

Let X be an ordered Banach space with semimonotone norm, $T(t)_{t\geq 0}$ a C_0-semigroup with generator A. If $A\colon X \supseteq \mathcal{D}(A) \to X$ is local and there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}\colon X \to \mathcal{D}(A) \subseteq X$ is local, then $T(t)_{t\geq 0}$ is local.

Applies to

$X = \mathrm{Pol}^2[0,1]$, $K = \{p \in X; p(x) \geq 0, x \in [0,1]\}$ is an order dense subspace of vector lattice $C[0,1]$. Let $q \in C([0,1))$ be bounded above. If $A\colon X \supseteq \mathcal{D}(A) \to X, x \mapsto qx$ is local and $(\lambda I - A)^{-1}$ is local for $\lambda > \sup_s q(s)$. Then $T(t)_{t\geq 0}$ is local.

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Does not apply to

X is asked to be complete with semimonotone norm. However, e.g.

- differential function space $\mathrm{C}^k(\Omega)$-spaces,
- Sobolev spaces $W^{p,q}(\Omega)$,

is not suitable.

Thank you!