The uo-dual of a Banach lattice

Foivos Xanthos

Department of Mathematics, Ryerson Univeristy, Toronto

Positivity IX, July 20 2017

Based on joint work with N. Gao and D. Leung

Roadmap

- Preliminaries
- The uo-dual and its relationship with the oc-dual
- Preduals of Banach lattices
- Applications to the dual representation problem of risk measures

Preliminaries

Throughout the presentation X denotes a Banach lattice.

Definition

A net (x_α) in X is said to **order converge** to $x \in X$, $x_\alpha \xrightarrow{o} x$, if $\exists$ another net (y_γ) s.t. $y_\gamma \downarrow 0$ and $\forall \gamma$, there exists α_0 such that $|x_\alpha - x| \leq y_\gamma$ for all $\alpha \geq \alpha_0$.

- A linear functional ϕ on X is said to be **order continuous** if $\phi(x_\alpha) \to 0$ for each $x_\alpha \xrightarrow{o} 0$
- $X_n^\sim$ is the space of order continuous functionals.

uo-convergence

Definition (Nakano, 1948)

A net (x_α) in X **unbounded order converges** to x, $x_\alpha \xrightarrow{uo} x$, if

$$|x_\alpha - x| \wedge y \xrightarrow{o} 0 \text{ for any } y \in X_+.$$

uo-continuous functionals

Definition

A linear functional ϕ on X is said **uo-continuous** if $\phi(x_\alpha) \to 0$ for each net (x_α) such that $x_\alpha \xrightarrow{uo} 0$.

uo-continuous functionals

Definition

A linear functional ϕ on X is said **uo-continuous** if $\phi(x_\alpha) \to 0$ for each net (x_α) such that $x_\alpha \xrightarrow{uo} 0$.

Proposition

Let X be a non-atomic Banach lattice. The only uo-continuous functional on X is 0.

uo-continuous functionals

Definition

A linear functional ϕ on X is said **uo-continuous** if $\phi(x_\alpha) \to 0$ for each net (x_α) such that $x_\alpha \xrightarrow{uo} 0$.

Proposition

Let X be a non-atomic Banach lattice. The only uo-continuous functional on X is 0.

Proof.

Let $\phi \neq 0$ be a non-zero uo-continuous functional of X and $x \in C_\phi, x > 0$. WLOG, $\phi > 0$. Since X is non-atomic, we can find an infinite disjoint sequence of non-zero vectors (x_n) in $[0, x]$.

uo-continuous functionals

Definition

A linear functional ϕ on X is said **uo-continuous** if $\phi(x_\alpha) \to 0$ for each net (x_α) such that $x_\alpha \xrightarrow{uo} 0$.

Proposition

Let X be a non-atomic Banach lattice. The only uo-continuous functional on X is 0.

Proof.

Let $\phi \neq 0$ be a non-zero uo-continuous functional of X and $x \in C_\phi, x > 0$. WLOG, $\phi > 0$. Since X is non-atomic, we can find an infinite disjoint sequence of non-zero vectors (x_n) in $[0, x]$. Then we have that $\phi(x_n) \neq 0$ and $y_n = \frac{x_n}{\phi(x_n)} \xrightarrow{uo} 0$.

uo-continuous functionals

Definition

A linear functional ϕ on X is said **uo-continuous** if $\phi(x_\alpha) \to 0$ for each net (x_α) such that $x_\alpha \xrightarrow{uo} 0$.

Proposition

Let X be a non-atomic Banach lattice. The only uo-continuous functional on X is 0.

Proof.

Let $\phi \neq 0$ be a non-zero uo-continuous functional of X and $x \in C_\phi, x > 0$. WLOG, $\phi > 0$. Since X is non-atomic, we can find an infinite disjoint sequence of non-zero vectors (x_n) in $[0, x]$. Then we have that $\phi(x_n) \neq 0$ and $y_n = \frac{x_n}{\phi(x_n)} \xrightarrow{uo} 0$. Thus $1 = \phi(y_n) \to 0$, a contradiction. $\square$

Boundedly uo-continuous functionals

Definition

A linear functional ϕ on X is said **boundedly uo-continuous** if $\phi(x_\alpha) \to 0$ whenever $\sup_\alpha \|x_\alpha\| < \infty$ and $x_\alpha \xrightarrow{uo} 0$.

Boundedly uo-continuous functionals

Definition

A linear functional ϕ on X is said **boundedly uo-continuous** if $\phi(x_\alpha) \to 0$ whenever $\sup_\alpha \|x_\alpha\| < \infty$ and $x_\alpha \xrightarrow{uo} 0$.

Remark

Since each order convergent net has a tail which is order bounded, and therefore, norm bounded, it is easy to see that every boundedly uo-continuous functional is order continuous.

Boundedly uo-continuous functionals

Definition

A linear functional ϕ on X is said **boundedly uo-continuous** if $\phi(x_\alpha) \to 0$ whenever $\sup_\alpha \|x_\alpha\| < \infty$ and $x_\alpha \xrightarrow{uo} 0$.

Remark

Since each order convergent net has a tail which is order bounded, and therefore, norm bounded, it is easy to see that every boundedly uo-continuous functional is order continuous.

Proposition

Let $\phi \in X_n^\sim$. TFAE:

- ϕ is boundedly uo-continuous
- $\phi(x_n) \to 0$ for any norm bounded uo-null sequence (x_n) in X.
- $\phi(x_n) \to 0$ for any norm bounded disjoint sequence (x_n) in X.

The uo-dual

Definition

$X_{uo}^{\sim}$ is the space of all boundedly uo-continuous functionals. We call it the **uo-dual** of X.

$$X_{uo}^{\sim} \subseteq X_n^{\sim} \subseteq X^*$$

Sequence spaces

X	$X_{uo}^{\sim}$	$X_n^{\sim}$
$\ell_p, 1 < p < \infty$	ℓ_q	ℓ_q
ℓ_1	c_0	ℓ_∞
ℓ_∞, c_0	ℓ_1	ℓ_1

Olricz spaces on $(\Omega, \mathcal{F}, \mathbb{P})$

X	$X_{uo}^{\sim}$	$X_n^{\sim}$
L_1	$\{0\}$	L_∞
L_∞	L_1	L_1
$L_\Phi \neq L_1, L_\infty$	H_Ψ	L_Ψ

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int \Phi\left(\frac{|f(\omega)|}{\lambda}\right) \mathrm{d}\mathbb{P} \leq 1\right\}$$

$$L_\Phi = \{f \in L_0 \mid ||f||_\Phi < +\infty\}, \Psi(t) = \sup\{st - \Phi(s) : s \geq 0\}$$

$$H_\Phi = \{f \in L_\Phi \mid \int \Phi\left(\frac{|f(\omega)|}{\lambda}\right) \mathrm{d}\mathbb{P} < \infty \ \forall \lambda > 0\}$$

How does $X_{uo}^{\sim}$ sit in $X_n^{\sim}$?

How does $X_{uo}^{\sim}$ sit in $X_n^{\sim}$?

Recall that the order continuous part of X is given by

$$X^a = \{x \in X \ : \ \text{every disjoint sequence in } [0, |x|] \text{ is norm null}\}$$

How does $X_{uo}^{\sim}$ sit in $X_n^{\sim}$?

Recall that the order continuous part of X is given by

$$X^a = \{x \in X \ : \ \text{every disjoint sequence in } [0, |x|] \text{ is norm null}\}$$

Theorem

$$X_{uo}^{\sim} = (X_n^{\sim})^a$$

That is, $X_{uo}^{\sim}$ is the largest norm closed ideal of $X_n^{\sim}$ which is order continuous in its own right.

When $X_{uo}^{\sim} = X^*$?

When $X_{uo}^{\sim} = X^*$?

(*): $X_n^{\sim} = X^*$ iff X has order continuous norm.

When $X_{uo}^{\sim} = X^*$?

(*): $X_n^{\sim} = X^*$ iff X has order continuous norm.

Corollary (Wickstead, 1977)

The following are equivalent:

(a) $X_{uo}^{\sim} = X^*$.

(b) *X and X^* have order continuous norm.*

(c) *Every norm bounded uo-null net is weakly null.*

When $X_{uo}^\sim = X^*$?

(*): $X_n^\sim = X^*$ iff X has order continuous norm.

Corollary (Wickstead, 1977)

The following are equivalent:

(a) $X_{uo}^\sim = X^*$.

(b) *X and X^* have order continuous norm.*

(c) *Every norm bounded uo-null net is weakly null.*

Proof.

Use $X_{uo}^\sim = (X_n^\sim)^a$ and (*) $\square$

When $(X^*)^\sim_{uo} = X$?

When $(X^*)^\sim_{uo} = X$?

(*): $(X^*)^\sim_n = X$ iff X is a KB-space

When $(X^*)^\sim_{uo} = X$?

(*): $(X^*)^\sim_n = X$ iff X is a KB-space

Theorem

The following are equivalent

(a) $(X^*)^\sim_{uo} = X$.

(b) X *has order continuous norm.*

When $(X^*)^\sim_{uo} = X$?

(*): $(X^*)^\sim_n = X$ iff X is a KB-space

Theorem

The following are equivalent

(a) $(X^*)^\sim_{uo} = X$.

(b) X *has order continuous norm.*

Sketch proof.

$(a) \Rightarrow (b)$: Apply $X^\sim_{uo} = (X^\sim_n)^a$

When $(X^*)^\sim_{uo} = X$?

(*): $(X^*)^\sim_n = X$ iff X is a KB-space

Theorem

The following are equivalent

(a) $(X^*)^\sim_{uo} = X$.

(b) X *has order continuous norm.*

Sketch proof.

$(a) \Rightarrow (b)$: Apply $X^\sim_{uo} = (X^\sim_n)^a$
$(b) \Rightarrow (a)$: Apply **Nakano's Theorem:** X *is order dense in* $(X^\sim_n)^\sim_n$

When $(X^*)^\sim_{uo} = X$?

(*): $(X^*)^\sim_n = X$ iff X is a KB-space

Theorem

The following are equivalent

(a) $(X^*)^\sim_{uo} = X$.

(b) X *has order continuous norm.*

Sketch proof.

$(a) \Rightarrow (b)$: Apply $X^\sim_{uo} = (X^\sim_n)^a$
$(b) \Rightarrow (a)$: Apply **Nakano's Theorem:** X *is order dense in* $(X^\sim_n)^\sim_n$ and $(X^*)^\sim_{uo} = ((X^*)^\sim_n)^a$,

$\square$

Preduals of Banach lattices

A dual Banach lattice need not have a unique up to lattice isomorphism Banach lattice predual (e.g. there exist non-atomic $C(K)$-spaces that are lattice isomorphic to ℓ_1 **Lacey, Wojtaszczyk 1976**)

Preduals of Banach lattices

A dual Banach lattice need not have a unique up to lattice isomorphism Banach lattice predual (e.g. there exist non-atomic $C(K)$-spaces that are lattice isomorphic to ℓ_1 **Lacey, Wojtaszczyk 1976**)

Corollary

Every Banach lattice X has at most one order continuous predual up to lattice isomorphism.

Preduals of Banach lattices

A dual Banach lattice need not have a unique up to lattice isomorphism Banach lattice predual (e.g. there exist non-atomic $C(K)$-spaces that are lattice isomorphic to ℓ_1 **Lacey, Wojtaszczyk 1976**)

Corollary

Every Banach lattice X has at most one order continuous predual up to lattice isomorphism.

If so, it is $X_{uo}^{\sim}$.

Preduals of Banach lattices

A dual Banach lattice need not have a unique up to lattice isomorphism Banach lattice predual (e.g. there exist non-atomic $C(K)$-spaces that are lattice isomorphic to ℓ_1 **Lacey, Wojtaszczyk 1976**)

Corollary

Every Banach lattice X has at most one order continuous predual up to lattice isomorphism.

If so, it is $X_{uo}^{\sim}$.

When X has an order continuous predual?

Monotonically complete Banach lattices

Definition

X is said to be **monotonically complete** if every norm bounded positive increasing net has a supremum.

Definition

X is said to be **boundedly uo-complete** if every norm bounded uo-Cauchy net is uo-convergent.

Monotonically complete Banach lattices

Definition

X is said to be **monotonically complete** if every norm bounded positive increasing net has a supremum.

Definition

X is said to be **boundedly uo-complete** if every norm bounded uo-Cauchy net is uo-convergent.

Theorem

Suppose $X_n^\sim$ separates the point of X. Then

$$X \text{ is monotonically complete} \iff X \text{ boundedly uo-complete.}$$

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Proof of $(a) \Rightarrow (b)$.

Apply Banach-Alaoglu.

$\square$

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(b) \Rightarrow (c)$.

It is suffices to show that X is boundedly uo-complete.

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(b) \Rightarrow (c)$.

It is suffices to show that X is boundedly uo-complete.

- Let (x_α) be a bounded uo-Cauchy net in B_X.

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(b) \Rightarrow (c)$.

It is suffices to show that X is boundedly uo-complete.

- Let (x_α) be a bounded uo-Cauchy net in B_X.
- Claim: X can be continuously embdedded into $(X_{uo}^{\sim})^*$ as a regular sublattice.

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(b) \Rightarrow (c)$.

It is suffices to show that X is boundedly uo-complete.

- Let (x_α) be a bounded uo-Cauchy net in B_X.
- Claim: X can be continuously embdedded into $(X_{uo}^{\sim})^*$ as a regular sublattice.
- *(Gao, Troitsky, X)*: (x_α) is a bounded uo-Cauchy in $(X_{uo}^{\sim})^*$.

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X then the following are equivalent

(a) *X has an order continuous predual.*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(b) \Rightarrow (c)$.

It is suffices to show that X is boundedly uo-complete.

- Let (x_α) be a bounded uo-Cauchy net in B_X.
- Claim: X can be continuously embdedded into $(X_{uo}^{\sim})^*$ as a regular sublattice.
- *(Gao, Troitsky, X)*: (x_α) is a bounded uo-Cauchy in $(X_{uo}^{\sim})^*$.
- (Gao): $x_\alpha \xrightarrow{uo,\sigma((X_{uo}^{\sim})^*, X_{uo}^{\sim})} x^{**}$ in $(X_{uo}^{\sim})^*$. By (b) we have that $x_\alpha \xrightarrow{uo} x^{**} \in X$.

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X, then the following are equivalent

(a) *X has an order continuous predual*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Banach lattices with order continuous predual

Theorem

If $X_{uo}^{\sim}$ separates the points of X, then the following are equivalent

(a) *X has an order continuous predual*

(b) *B_X is relatively $\sigma(X, X_{uo}^{\sim})$-compact in X.*

(c) *X is monotonically complete.*

Sketch Proof of $(c) \Rightarrow (a)$.

$$
\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "i"'] & (X_n^{\sim})_n^{\sim} \arrow[dl, "R"] \\
(X_{uo}^{\sim})_n^{\sim} = (X_{uo}^{\sim})^* &
\end{tikzcd}
$$

- (**Meyer-Nieberg**): j *is a lattice isomorphism.*
- Claim: The restriction map R of j is a lattice isometry from $(X_n^{\sim})_n^{\sim}$ onto $(X_{uo}^{\sim})^*$.
- $i = Rj$ is a (surjective) lattice isomorphism.

$\square$

Delbaen's representation Theorem of risk measures on L_∞

Theorem (2001)

For any proper convex functional $\rho : L_\infty(\mathbb{P}) \to (-\infty, \infty]$, the following statements are equivalent:

1. $\rho(x) = \sup_{y \in L_1(\mathbb{P})}(\langle x, y\rangle - \rho^*(y))$ *for any* $x \in L_\infty(\mathbb{P})$, *where* $\rho^*(y) = \sup_{x \in L_\infty(\mathbb{P})}(\langle x, y\rangle - \rho(x))$ *for any* $y \in L_1(\mathbb{P})$,
2. $\rho(x) \leq \liminf_n \rho(x_n)$ *for any* <u>*bounded*</u> *sequence* (x_n) *in* $L_\infty(\mathbb{P})$ *converging a.e. to* x.

Delbaen's representation Theorem of risk measures on L_∞

Theorem (2001)

For any proper convex functional $\rho : L_\infty(\mathbb{P}) \to (-\infty, \infty]$, the following statements are equivalent:

1. $\rho(x) = \sup_{y \in L_1(\mathbb{P})}(\langle x, y\rangle - \rho^*(y))$ *for any* $x \in L_\infty(\mathbb{P})$, *where* $\rho^*(y) = \sup_{x \in L_\infty(\mathbb{P})}(\langle x, y\rangle - \rho(x))$ *for any* $y \in L_1(\mathbb{P})$,
2. $\rho(x) \leq \liminf_n \rho(x_n)$ *for any <u>bounded</u> sequence* (x_n) *in* $L_\infty(\mathbb{P})$ *converging a.e. to* x.

Problem (Dual representation problem)

Generalize the above result to Banach lattices. (see work of Biagini, Cheredito, Delbaen, Frittelli, Orihuela, Owari, Schachermayer)

Delbaen's representation Theorem of risk measures on L_∞

Theorem (2001)

For any proper convex functional $\rho : L_\infty(\mathbb{P}) \to (-\infty, \infty]$, the following statements are equivalent:

1. $\rho(x) = \sup_{y \in L_1(\mathbb{P})}(\langle x, y\rangle - \rho^*(y))$ *for any* $x \in L_\infty(\mathbb{P})$, *where* $\rho^*(y) = \sup_{x \in L_\infty(\mathbb{P})}(\langle x, y\rangle - \rho(x))$ *for any* $y \in L_1(\mathbb{P})$,
2. $\rho(x) \leq \liminf_n \rho(x_n)$ *for any* <u>*bounded*</u> *sequence* (x_n) *in* $L_\infty(\mathbb{P})$ *converging a.e. to* x.

Problem (Dual representation problem)

Generalize the above result to Banach lattices. (see work of Biagini, Cheredito, Delbaen, Frittelli, Orihuela, Owari, Schachermayer)

How we can interpret the continuity condition in (2) if ρ acts on a Banach lattice?

o-approach

$$\rho(x) \leq \liminf_n \rho(x_n) \text{ whenever } x_n \xrightarrow{o} x$$

.

o-approach

$$\rho(x) \leq \liminf_n \rho(x_n) \text{ whenever } x_n \xrightarrow{o} x$$

.

In this case the dual representation problem can be reduced to the following one

Problem (Owari, 2014)

Is every order closed convex set $\sigma(X, X_n^\sim)$-closed?

o-approach

$$\rho(x) \leq \liminf_n \rho(x_n) \text{ whenever } x_n \xrightarrow{o} x$$

.

In this case the dual representation problem can be reduced to the following one

Problem (Owari, 2014)

Is every order closed convex set $\sigma(X, X_n^\sim)$-closed?

Denny's talk: There exists an order closed set in an Orlicz space X that is NOT $\sigma(X, X_n^\sim)$-closed.

uo-approach

$$\rho(x) \leq \liminf_n \rho(x_n) \text{ whenever } x_n \xrightarrow{uo} x \text{ and } \sup_{n\in\mathbb{N}} ||x_n|| < +\infty.$$

uo-approach

$$\rho(x) \le \liminf_n \rho(x_n) \text{ whenever } x_n \xrightarrow{uo} x \text{ and } \sup_{n\in\mathbb{N}} ||x_n|| < +\infty.$$

In this case the dual representation problem can be reduced to the following one

Problem

Is every boundedly uo-closed convex set $\sigma(X, X_{uo}^{\sim})$-closed?

Definition

A set $C \subseteq X$ is said to be boundedly uo- closed if $C = \overline{C}^{buo}$, where

$$\overline{C}^{buo} := \{x \in X : \; x_\alpha \xrightarrow{uo} x \text{ for some bounded net } (x_\alpha) \text{ in } C\}.$$

Proposition

Let X be a σ-order complete Banach lattice. The following statements are equivalent.

1. $\overline{C}^{buo} = \overline{C}^{\sigma(X, X_{uo}^{\sim})}$ for every convex set C.
2. X and X^* are both order continuous.

Proposition

Let X be a σ-order complete Banach lattice. The following statements are equivalent.

1. $\overline{C}^{buo} = \overline{C}^{\sigma(X, X_{uo}^{\sim})}$ for every convex set C.
2. X and X^* are both order continuous.

The proof of (1) $\Rightarrow$ (2) is based on the following result due to **Ostrovskii**: *There exist a subspace W of ℓ^∞ and $w \in \overline{W}^{\sigma(\ell^\infty, \ell^1)}$ such that w is not the $\sigma(\ell^\infty, \ell^1)$-limit of any sequence in W.*

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^{\sim})^*$.

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^{\sim})^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^{\sim})$-closed (Krein-Smulian Theorem)

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^\sim)$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^\sim)^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^\sim)$-closed (Krein-Smulian Theorem)
- $\widetilde{C}$ is $|\sigma|(X, X_{uo}^\sim)$ (Kaplan's Theorem)

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^{\sim})^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^{\sim})$-closed (Krein-Smulian Theorem)
- $\widetilde{C}$ is $|\sigma|(X, X_{uo}^{\sim})$ (Kaplan's Theorem)

Since X has strictly positive bounded uo-continuous functional ϕ, we can embed X into the norm completion $\widetilde{X}$ of $(X, \phi(|\cdot|))$ as a regular sublattice.

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^\sim)$-closed.

Sketch Proof of Theorem.

By the uniquness of the order continuous predual we have that $X = (X_{uo}^\sim)^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^\sim)$-closed (Krein-Smulian Theorem)
- $\widetilde{C}$ is $|\sigma|(X, X_{uo}^\sim)$ (Kaplan's Theorem)

Since X has strictly positive bounded uo-continuous functional ϕ, we can embed X into the norm completion $\widetilde{X}$ of $(X, \phi(|\cdot|))$ as a regular sublattice. Let $x \in \overline{\widetilde{C}}^{|\sigma|(X, X_{uo}^\sim)}$ and pick $(x_a) \subseteq \widetilde{C}$ such that $x_\alpha \xrightarrow{|\sigma|(X, X_{uo}^\sim)} x$.

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^{\sim})^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^{\sim})$-closed (Krein-Smulian Theorem)
- $\widetilde{C}$ is $|\sigma|(X, X_{uo}^{\sim})$ (Kaplan's Theorem)

Since X has strictly positive bounded uo-continuous functional ϕ, we can embed X into the norm completion $\widetilde{X}$ of $(X, \phi(|\cdot|))$ as a regular sublattice. Let $x \in \overline{\widetilde{C}}^{|\sigma|(X, X_{uo}^{\sim})}$ and pick $(x_a) \subseteq \widetilde{C}$ such that $x_\alpha \xrightarrow{|\sigma|(X, X_{uo}^{\sim})} x$. Then we can extract a sequence (x_n) from (x_α) such that $x_n \xrightarrow{o} x$ in $\widetilde{X}$.

Theorem

Let Y be an order continuous Banach lattice with weak units, and let $X = Y^*$. Then every boundedly uo-closed convex set C in X is $\sigma(X, X_{uo}^{\sim})$-closed.

Sketch Proof of Theorem.

By the uniqness of the order continuous predual we have that $X = (X_{uo}^{\sim})^*$. Our problem can be reduced to the following ones: Show that

- $\widetilde{C} = C \cap kB_X$ is $\sigma(X, X_{uo}^{\sim})$-closed (Krein-Smulian Theorem)
- $\widetilde{C}$ is $|\sigma|(X, X_{uo}^{\sim})$ (Kaplan's Theorem)

Since X has strictly positive bounded uo-continuous functional ϕ, we can embed X into the norm completion $\widetilde{X}$ of $(X, \phi(|\cdot|))$ as a regular sublattice. Let $x \in \overline{\widetilde{C}}^{|\sigma|(X, X_{uo}^{\sim})}$ and pick $(x_a) \subseteq \widetilde{C}$ such that $x_\alpha \xrightarrow{|\sigma|(X, X_{uo}^{\sim})} x$. Then we can extract a sequence (x_n) from (x_α) such that $x_n \xrightarrow{o} x$ in $\widetilde{X}$. Since X is a regular sublattice we have that $x_n \xrightarrow{uo} x$ in X and thus $x \in \widetilde{C}$. □

uo-dual representations of risk measures on Orlicz spaces

Theorem (Gao and X, Mathematical Finance)

If an Orlicz space L_Φ is not equal to L_1, then the following statements are equivalent for every proper (i.e., not identically ∞) convex functional $\rho : L_\Phi \to (-\infty, \infty]$.

1. $\rho(f) = \sup_{g \in H_\Psi} \left(\int fg - \rho^*(g) \right)$ *for any* $f \in L_\Phi$, *where* H_Ψ *is the conjugate Orlicz heart, and*
   $\rho^*(g) = \sup_{f \in L_\Phi} \left(\int fg - \rho(f) \right)$ *for any* $g \in H_\Psi$.
2. $\rho(f) \leq \liminf_n \rho(f_n)$, *whenever* $f_n \xrightarrow{a.e.} f$ *and* (f_n) *is norm bounded in* L_Φ.

Thank you very much for your attention!