Density of cones and the projective tensor product

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Throughout this talk:

- (X, C, τ) vector space, equipped with a cone C and a linear Hausdorff topology τ
- $W \subset X$ wedge: $W + W = W$, $\mathbb{R}_+ W = W$
- $C \subset X$ cone: wedge with $C \cap -C = \{0\}$
- X': linear functionals
- X^*: continuous linear functionals
- X'_+ (X^*_+): (continuous) positive linear functionals

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Q: When is a cone C dense in X?

Lemma

If τ is locally convex, then W is dense in X if and only if $X_+^* = \{0\}$

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Proof.

Suppose W is dense in X and let $\phi \in X^*_+$. Then $\phi(W) \subset \mathbb{R}_+$, so $\phi(X) \subset \mathbb{R}_+$, so $\phi = 0$.
Suppose W is not dense in X. Let $x \notin \overline{W}$ and let $\phi \in X^*$ be such that

$$\phi(x) < \lambda < \phi(\overline{W}).$$

Then $0 \neq \phi \in X^*_+$. $\square$

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Lemma

If $\dim(X) < \infty$ *and W is dense in X, then $W = X$.*

Proof.

Approximate each $x \in \operatorname{ex} B_{\ell_n^\infty}$ by elements $y \in W$ and collect these y in a set $S \subset W$. Since

$$0 \in (\operatorname{co} \operatorname{ex} B_{\ell_n^\infty})^\circ,$$

$0 \in (\operatorname{co} S)^\circ \subset W^\circ$, so $W = X$. □

$L^p[0,1]$

- $0 < p < 1$
- $X = L^p[0,1] = \{f\colon \int |f|^p < \infty\}$
- $C = L^p[0,1]_+$
- $d(f,g) := \int |f-g|^p$
- (X,d) complete metric space with $X^* = \{0\}$
- If $\phi \in X'_+$ then $\phi \in X^*$, so $X'_+ = \{0\}$
- $X^*_+ = \{0\}$ for *any* τ
- C is dense in X for any locally convex τ
- However, C is d-closed!

Proposition

$X'_+ = \{0\}$ *if and only if* C *is dense in* X *for some (any) locally convex* τ *with* $X^* = X'$

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Examples of such τ:

- $\sigma(X, X')$
- Strongest locally convex topology: all seminorms / all absolutely convex absorbing subsets

Lexicographic cones

- I partially ordered index set
- $F_0(I) := \{f : I \to \mathbb{R} \colon \operatorname{supp}(f) < \infty\}$
- $C := \{f \in F_0(I) \colon f(j) < 0 \Rightarrow \exists i < j \colon f(i) > 0\}$
- $\operatorname{Lex}(I) := (F_0(I), C)$
- If I is totally ordered, then

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Examples:

- If $I = \{1, \ldots, n\}$, $1 < \ldots < n$, then $C = \mathbb{R}^n_{lex}$
- If $I = \{1, \ldots, n\}$ with the trivial order, then $C = \mathbb{R}^n_+$

$$C := \{f \in F_0(I) \colon f(j) < 0 \Rightarrow \exists i < j \colon f(i) > 0\}$$

Properties:

- If I is totally ordered, then $\mathrm{Lex}(I)$ is totally ordered
- If $I = J \sqcup K$ with J and K incomparable, then

$$\mathrm{Lex}(I) \cong \mathrm{Lex}(J) \oplus \mathrm{Lex}(K)$$

- C is Archimedean if and only if I has the trivial order
- ...

$\mathrm{Lex}(\mathbb{Z})$

Theorem

C is dense in $\mathrm{Lex}(\mathbb{Z})$ for any τ.

Proof.

Let $f \in \mathrm{Lex}(\mathbb{Z})$ and let $k < \min(\mathrm{supp}(f))$. Then $C \ni f + e_k/n \to f$. $\square$

Corollary

$\mathrm{Lex}(\mathbb{Z})'_+ = \{0\}$

Projective tensor product

- (X, X_+), (Y, Y_+), (Z, Z_+) ordered vector spaces

Definition

The *projective cone* $C(X_+, Y_+) \subset X \otimes Y$ is defined by

$$C(X_+, Y_+) := \left\{ \sum_{i=1}^{n} x_i \otimes y_i \colon x_i \in X_+, y_i \in Y_+ \right\}.$$

The ordered vector space $(X \otimes Y, C(X_+, Y_+))$ is called the *projective tensor product* of X and Y.

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Proposition (universal property)

If $\phi \colon X \times Y \to Z$ is a positive bilinear map, then there exists a unique positive linear map $\tilde{\phi} \colon X \otimes Y \to Z$ such that $\phi = \tilde{\phi} \circ \otimes$.

Question: is the projective cone a cone?

Theorem

Let I and J be partially ordered index sets. Then

$$\mathrm{Lex}(I) \otimes \mathrm{Lex}(J) \cong \mathrm{Lex}(I \times J).$$

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Theorem

Every cone $C \subset \mathbb{R}^n$ is contained in a cone isomorphic to $\mathbb{R}^n_{lex}$.

Theorem

The projective cone is a cone.

Proof.

Let $\pm u \in C(X_+, Y_+)$. Then $\pm u \in E \otimes F$ with $\dim(E), \dim(F) < \infty$. So $\pm u \in C(E_+, F_+) \subset C(\mathbb{R}^n_{lex}, \mathbb{R}^m_{lex})$ which is isomorphic to the cone of $\mathrm{Lex}(\{1, \ldots, n\} \times \{1, \ldots, m\})$. Hence $u = 0$. $\square$

Thank you for your attention!

Questions?