Weak containment by restrictions of induced representations

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Containment of Representations

Let π and σ be representations of a locally compact group G.

Three notions for what it means for π to contain σ:

- σ is unitarily equivalent to a subrepresentation of π
- σ is quasi-contained in π
- σ is weakly contained in π

Quasi-containment

π, σ – representations of G
$\mathrm{VN}_\pi := \pi(G)'' \subset \mathcal{B}(\mathcal{H}_\pi)$

Definition

σ is *quasi-contained* in π is σ is unitarily equivalent to subrepresentation of some amplification of π.

Theorem

π quasi-contains σ iff the identity map on G extends to a normal $*$-homomorphism $\mathrm{VN}_\pi \to \mathrm{VN}_\sigma$.

Weak containment

π, σ – representations of G
$\pi_{\xi,\eta}\colon G \to \mathbb{C}$ defined by $\pi_{\xi,\eta}(s) = \langle \pi(s)\xi, \eta\rangle$ for $\xi, \eta \in \mathcal{H}_\pi$
$\mathrm{C}^*_\pi := \overline{\pi(\mathrm{L}^1(G))}^{\|\cdot\|}$

Definition

σ is *weakly contained* in π (write $\sigma \prec \pi$) if for every $\xi \in \mathcal{H}_\sigma$, $\sigma_{\xi,\xi}$ is the limit of positive definite functions of the form $\sum_{i=1}^{N} \pi_{\eta_i,\eta_i}$ in the topology of uniform convergence on compact subsets of G.

Theorem

π weakly contains σ if and only if the identity map on $\mathrm{L}^1(G)$ extends to $*$-homomorphism $\mathrm{C}^*_\pi \to \mathrm{C}^*_\sigma$.

Main problem

Let H be a closed subgroup of a locally compact group G and π a representation of H.

When is π "contained" in $(\mathrm{Ind}_H^G \pi)|_H$?

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Easy exercise

If G be a discrete group, then π is unitarily equivalent to a subrepresentation of $(\mathrm{Ind}_H^G \pi)|_H$.

Aside: Classes of Locally Compact Groups

$\tau : G \to \mathcal{B}(\mathrm{L}^1(G))$ defined by $\tau(s)f(t) = f(s^{-1}ts)\Delta(s)$

Definition

A locally compact group G is a *SIN* group if the identity of G admits a neighbhourhood base consisting of conjugation invariant compact sets K, i.e., sets K such that $s^{-1}Ks = K$ for all $s \in G$.

Example

- Abelian groups,
- Discrete groups,
- Compact groups

Theorem (Mosak)

A locally compact group G is SIN if and only if $\mathrm{L}^1(G)$ has a *central BAI*, i.e., a BAI $\{e_\alpha\} \subset \mathrm{L}^1(G)$ such that $\tau(s)e_\alpha = e_\alpha$ for all $s \in G$.

Aside: Classes of Locally Compact Groups

Definition

A locally compact group G is *QSIN* if $\mathrm{L}^1(G)$ has a *quasi-central BAI*, i.e., a BAI $\{e_\alpha\} \subset \mathrm{L}^1(G)$ such that $\|\tau(s)e_\alpha - e_\alpha\| \to 0$ uniformly on compact subsets of G.

Theorem (Losert-Rindler)

Every amenable group is QSIN.

Back to main question

Question

When does $(\mathrm{Ind}_H^G \pi)|_H$ contain π?

Theorem (Cowling-Rodway)

Let G be a SIN group. Then $(\mathrm{Ind}_H^G \pi)|_H$ quasi-contains π for every closed subgroup H of G and representation π of H.

Back to main question

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When does $(\mathrm{Ind}_H^G \pi)|_H$ contain π?

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Let G be a SIN group. Then $(\mathrm{Ind}_H^G \pi)|_H$ quasi-contains π for every closed subgroup H of G and representation π of H.

Example (Khalil)

The above result fails for $G = \mathbb{R} \rtimes \mathbb{R}^+$ be the $ax + b$ group and H be the subgroup $\mathbb{R}$.

Main result

Theorem (W.)

Let G be a QSIN group. Then $\pi \prec (\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\pi)|_H$ for every closed subgroup $H \leq G$ and representation π of H.

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Example (Bekka)

The above result fails for $G = \mathrm{SL}(2, \mathbb{R})$ and $H = \mathrm{SL}(2, \mathbb{Z})$.

Completely Positive Maps

Definition

Let A and B be C*-algebras.
A linear map $\phi\colon A \to B$ is *completely positive* if

$$[a_{ij}] \in M_n(A) \text{ is positive} \Rightarrow [\phi(a_{ij})] \in M_n(B) \text{ is positive.}$$

Nuclear C*-algebras

Definition

A C*-algebra A is *nuclear* if $A \otimes_{\min} B = A \otimes_{\max} B$ for every C*-algebra B.

Definition

A C*-algebra A has the *completely positive approximation property* (*CPAP*) if there exist ccp maps $\varphi_i : A \to \mathrm{M}_{n_i}(\mathbb{C})$ and $\psi_i : \mathrm{M}_{n_i}(\mathbb{C}) \to A$ such that

$$\|\psi_i \circ \phi_i(a) - a\| \to 0$$

for every $a \in A$.

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for every $a \in A$.

Theorem(Kirchberg)

A C*-algebra A is nuclear iff it has the CPAP.

Nuclearity of Group C*-algebras

$\mathrm{C}^*_{\mathrm{r}}(G) := \mathrm{C}^*_{\lambda} = \overline{\lambda(\mathrm{L}^1(G))}^{\|\cdot\|}$
$\mathrm{C}^*(G) := \mathrm{C}^*_{\pi_u}$, where π_u is universal representation of G

Theorem (Lance)

Let G be a discrete group. Then G is amenable if and only if $\mathrm{C}^*_{\mathrm{r}}(G)$ is nuclear.

Theorem (Connes)

Let G be a separable and connected. Then $\mathrm{C}^*(G)$ is nuclear.

Exact C*-algebras

Definition

A C*-algebra A is *exact* if for every short exact sequence $0 \to J \to B \to C \to 0$ of C*-algebras, the sequence

$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} C \to 0$$

is exact.

Theorem (Kirchberg)

Let A be a C*-algebra and suppose that $A \hookrightarrow \mathcal{B}(\mathcal{H})$ is a faithful embedding. The C*-algebra A has is exact if and only if there exists ccp maps $\varphi_i : A \to M_{n_i}(\mathbb{C})$ and $\psi_i : M_{n_i}(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ such that $\|\psi_i \circ \varphi_i(a) - a\| \to 0$ for all $a \in A$.

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$$\text{Nuclear} \Rightarrow \text{Exact}$$

$\mathrm{C}^*(\mathbb{F}_2)$ is not exact

Theorem (Wasserman)

The sequence

$$0 \to \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{J} \to \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_2) \to \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2) \to 0$$

is not exact, where J is the kernel of $\mathrm{C}^*(\mathbb{F}_2) \to \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)$.

Local properties of C*-algebras

Local reflexivity and the *local lifting property* (LLP) are C*-algebraic properties which are weaker than nuclearity.

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Definition

A unital C^*-algebra A has the *LLP* if any ucp map $\varphi : A \to B/J$ is locally liftable, i.e., for any finite dimensional operator system $E \subset A$, there exists a ucp map $\psi : E \to B$ such that $\varphi = q \circ \psi$ (where $q : B \to B/J$ is the quotient map).
A nonuital C*-algebra A is said to have the LLP if its unitization does.

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Theorem (Kirchberg)

A C*-algebra A has the LLP if and only if $A \otimes_{\min} \mathcal{B}(\mathcal{H}) = A \otimes_{\max} \mathcal{B}(\mathcal{H})$ canonically.

Local properties of C*-algebras

Theorem (Effros-Haagerup)

If A is a locally reflexive C*-algebra, then the sequence

$$0 \to J \otimes_{\min} C \to A \otimes_{\min} C \to A/J \otimes_{\min} C \to 0$$

is exact for every closed two-sided ideal J of A and every C*-algebra C.

Theorem (Effros-Haagerup)

Let B be a C*-algebra and J a closed two sided ideal of B. If $A := B/J$ has the local lifting property, then the sequence

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to A \otimes_{\min} C \to 0$$

is exact for every C*-algebra C.

Local properties of group C*-algebras

Theorem (Wasserman)

The sequence

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is not exact, where J is the kernel of $\mathrm{C}^*(\mathbb{F}_2) \to \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)$.

Corollary

$\mathrm{C}^*(\mathbb{F}_2)$ is not locally reflexive and $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)$ does not have the LLP.

Local properties of group C*-algebras

Theorem (W.)

Let G be a QSIN group which contains $\mathbb{F}_2$ as a closed subgroup. Then

$$0 \to \mathrm{C}^*(G) \otimes_{\mathrm{min}} \mathrm{K} \to \mathrm{C}^*(G) \otimes_{\mathrm{min}} \mathrm{C}^*(G) \to \mathrm{C}^*(G) \otimes_{\mathrm{min}} \mathrm{C}^*_{\mathrm{r}}(G) \to 0$$

is not exact, where K is the kernel of $\mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{r}}(G)$.

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is not exact, where K is the kernel of $\mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{r}}(G)$.

Key Fact: $(\mathrm{Ind}_{F_2 \times \mathbb{F}_2}^{G \times G} \pi)|_{\mathbb{F}_2 \times \mathbb{F}_2}$ weakly contains π for every representation π of $\mathbb{F}_2 \times \mathbb{F}_2$.

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Corollary

If G is QSIN and contains $\mathbb{F}_2$ as a closed subgroup, then $\mathrm{C}^*(G)$ is not locally reflexive and $\mathrm{C}^*_{\mathrm{r}}(G)$ does not have the LLP.

Thank you!