The Riesz-Kantorovich Formulae

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Michael Elliott completed a Ph.D. with me in 2001. Most of the results that I will talk about today are his, but he is unable to be here today and has given me permission to talk about his work.

The Riesz-Kantorovich Formulae

It has been known since the inception of our subject that if F is a Dedekind complete Riesz space and the linear operator $T : E \to F$ is order bounded then T has a modulus given, on E_+, by the formula

$$|T|(x) = \sup\{Ty : |y| \leq x\}.$$

This, and similar formulae for T^+ and T^-, are the *Riesz-Kantorovich formulae*. There are other circumstance when it is known that an order bounded operator has a modulus given by the Riesz-Kantorovich formula. For example, in the Banach lattice setting this is true for all order bounded operators if:

1. E is atomic with an order continuous norm.
2. E has separable order intervals and F is Dedekind σ-complete.

The Riesz-Kantorovich Formulae

These are conditions that guarantee that *every* order bounded operator has a modulus and therefore that the order bounded and regular operators coincide and that they form a Riesz space with the lattice operations given the the *Riesz-Kantorovich formulae*.

Whether or not the modulus of an operator has to be given by the Riesz-Kantorovich formula has been an outstanding question for many years, even though it has rarely been explicitly posed. Looking back, I think that there are really two questions to ask about operators between Riesz spaces.

The Riesz-Kantorovich Formulae

1. If $\mathcal{L}^r(E,F)$ is a Riesz space, are the lattice operations given by the Riesz-Kantorovich formulae?
2. If $T \in \mathcal{L}^r(E,F)$ has a modulus, is that modulus given by the Riesz-Kantorovich formula?

In the remainder of this talk, I will survey what I know about the answer to these two separate questions in the Banach lattice setting.

The Lattice Problem

An important step towards at least a partial solution of the first problem was given by van Rooij in 1984.

A Banach lattice E has *property* $(\star)$ if for every sequence (f_n) in E^*_+ which converges $\sigma(E^*, E)$ to $f \in E^*_+$ we have $|f_n - f| \to 0$ for $\sigma(E^*, E)$.

Theorem (van Rooij, 1984)

If E and F are Banach lattices such that $\mathcal{L}^r(E, F)$ is a Riesz space then either E has property $(\star)$ or F is Dedekind σ-complete.

The Lattice Problem

Property $(\star)$ is not very intuitive, but in many cases there is a simpler description. In particular:

Theorem (Chen Zili & W., 1999)

If E is a separable Banach lattice then E has property $(\star)$ if and only if E is atomic with an order continuous norm.

The Lattice Problem

Putting these two results together with the cases cited in the previous section, we have:

Theorem

If E and F are Banach lattices and E is separable, then $\mathcal{L}^r(E, F)$ is a Riesz space if and only if either

1. *E is atomic with an order continuous norm, or*
2. *F is Dedekind σ-complete.*

In particular, the lattice operations are given by the Riesz-Kantorovich formulae.

The observant among you will note the gap between the conditions that E is separable and that it have separable order intervals. So far I have been unable to prove that a Banach lattice with separable order intervals and property $(\star)$ must be

The Lattice Problem

What can we say if we get away from separability? It turns out that we can say something if we assume order continuity of the norm in E. The key to being able to obtain necessary and sufficient conditions is to be able to work with Banach lattices where *every* non-trivial order interval has the same density characteristic (i.e. the smallest cardinal of a dense subset.) We call such Banach lattices *homogeneous*. If the density character of order intervals in a homogeneous Banach lattice E is $\mathfrak{a}$ then we say that E is $\mathfrak{a}$-homogeneous.

The Lattice Problem

Banach lattices with an order continuous norm can be decomposed into sums of homogeneous Banach lattices.

Theorem (Elliott, 2001)

Let E be a Banach lattice with an order continuous norm. There is a unique ordinal τ, a cofinal subset Σ of τ and a pairwise disjoint collection $(E_\sigma)_{\sigma\in\Sigma}$ of bands in E such that E_σ is $\aleph_\sigma$-homogeneous and $E = \mathrm{at}(E) \oplus \sum_{\sigma\in\Sigma} E_\sigma$, where $at(E)$ is the band generated by the atoms in E.

The Lattice Problem

Elliott originally gave an ingenious, but technically difficult, abstract proof of this theorem. A simpler way to understand it is to use the well-known embedding of a Banach lattice E with an weak order unit and an order continuous norm between $L_\infty(\mu)$ and $L_1(\mu)$. In the case that there is no weak order unit, take a maximal disjoint family in E_+, carry out this embedding for each generated ideal and embed E into the (uncountable) ℓ_1 sum of these embeddings. Given a result of Amemiya that the order continuous norm and the L_1 norms generate the same topology on order intervals, this reduces the proof to the case $E = L_1(\mu)$. That follows easily from Maharam's representation of abstract L-spaces, which is nowhere near as well known as Kakutani's even though it is potentially much more useful.

The Lattice Problem

In the statement of the following result, $\mathbf{2} = \{0, 1\}$, γ is the measure on $\mathbf{2}$ with $\gamma(0) = \gamma(1) = \frac{1}{2}$, and $\mathfrak{a}F$ denotes the ℓ_1-direct sum of $\mathfrak{a}$ many copies of F. It is routine to show that $L_1(\mathbf{2}^{\mathfrak{a}}, \gamma^{\mathfrak{a}})$ is $\mathfrak{a}$-homogeneous.

Theorem (Maharam, 1942)

Let E be an AL-space. There exists a unique well-ordered family $(\mathfrak{a}_\sigma)_{-1 \leq \sigma < \tau}$ such that:

1. *for each $\sigma \geq 0$, each $\mathfrak{a}_\sigma$ is equal to 0, or to 1, or is uncountable.*
2. $\{\sigma : \mathfrak{a}_\sigma \neq 0\}$ *is cofinal in τ, and*
3. *Y is isometrically order isomorphic to $\ell_1(\mathfrak{a}_{-1}) \oplus_1 \ell_1\big(\mathfrak{a}_\sigma L_1(\mathbf{2}^{\aleph_\sigma}, \gamma^{\aleph_\sigma}); 0 \leq \sigma < \tau\big)$.*

The Lattice Problem

The fact that the decomposition of a Banach lattice with an order continuous norm is indexed by a family defined by a strict inequality cannot be avoided, nor can the subsequent complication in the statement of the main result coming up.

If $\mathfrak{a}$ is an infinite cardinal, we say that a Banach lattice F is $\mathfrak{a}$-*complete* if every non-empty set, of cardinality at most $\mathfrak{a}$ which is bounded above, has a supremum. We say that F is $<\mathfrak{a}$-*complete* if every non-empty set, of cardinality strictly less than $\mathfrak{a}$ which is bounded above, has a supremum.

The Lattice Problem

Theorem (Elliott, 2001)

Let E be a Banach lattice with an order continuous norm and F be any Banach lattice. $\mathcal{L}^r(E,F)$ is a Riesz space if and only if either

1. *E is atomic with an order continuous norm or*
2. *F is Dedekind $<\mathfrak{a}$-complete where $\mathfrak{a}$ is the smallest cardinal that is greater than the density character of every order interval in E.*

In particular, the lattice operations are given by the Riesz-Kantorovich formulae.

An uncountable cardinal is *weakly inaccessible* if it is a regular limit cardinal. Their existence cannot be proved within ZFC.

The Lattice Problem

Theorem (Elliott, 2001)

Assume there are no weakly inaccessible cardinals. Let E be a Banach lattice with an order continuous norm and F be any Banach lattice. $\mathcal{L}^r(E, F)$ is a Riesz space if and only if either

1. *E is atomic with an order continuous norm or*
2. *F is Dedekind $\mathfrak{a}$-complete where $\mathfrak{a}$ is the smallest cardinal that is greater than or equal to the density character of every order interval in E.*

In particular, the lattice operations are given by the Riesz-Kantorovich formulae.

Elliott also gave an example to show that the preceding result is false if there is a weakly inaccessible cardinal.

The Individual Operator Problem

In October 2014, Michael sent me an counterexample to the individual operator problem. I, and several other people, have checked the proof carefully and I have rewritten the proof in my own terms to convince myself of its correctness. I will describe the example, but refrain through lack of time, from going into details of the proofs. The surprising feature of the example, to my mind, is that is not at all that exotic. The domain is $L_1([0,1])$, the range space is a $C(K)$ and the operator is constructed in a familiar fashion.

The Individual Operator Problem

Definition

If A is a non-empty set and $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ then A^n will denote the set of all n-tuples from A and we set $\mathbb{T}(A) = \bigcup_{n=0}^{\infty} A^n$. When it is necessary to refer to the empty tuple $()$, we use the notation $\emptyset$. If $\tau \in \mathbb{T}(A)$, we write $\tau = (\tau_0, \tau_1, \ldots, \tau_{n-1})$ and call n the *length* of τ. If $\sigma, \tau \in \mathbb{T}(A)$, with lengths m and n respectively, then we set $\sigma \oplus \tau = (\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1})$.

If $n \in \mathbb{N}^*$ then $\mathbf{3}^n = \{0, 1, 2\}^n$ has precisely 3^n elements so there is a bijection ϕ_n between $\{0, 1, \ldots, 3^n - 1\}$ and $\mathbf{3}^n$. Using any, arbitrary but fixed, choice of ϕ_n we may define $\Phi_n : \mathbb{N}^* \to \mathbf{3}^n \subset \mathbb{T}(3)$ by $\Phi_n(k) = \phi_n(k \mod 3^n)$, so that $\Phi_n(k)$ cycles through the elements of $\mathbf{3}^n$.

The Individual Operator Problem

The range of the operators in Elliott's example is a closed sublattice of $\ell_\infty\big(\mathbb{T}(\mathbb{N}^*)\big)$. We write the elements of this space as $x = (x_\tau)$ where $\tau \in \mathbb{T}(\mathbb{N}^*)$. Define

$$F = \{x = (x_\tau) : \lim_{k\to\infty} x_{\tau\oplus\{k\}} = x_\tau \forall \tau \in \mathbb{T}(\mathbb{N}^*)\}.$$

It will be clear that F is a sublattice of $\ell_\infty\big(\mathbb{T}(\mathbb{N}^*)\big)$ containing the constants, which is closed under the supremum norm, so that with the supremum norm F is an AM-space with an order unit and is hence isometrically order isomorphic to $C(K)$ for some compact Hausdorff space K.

The Individual Operator Problem

The domain of the operators in Elliott's example is just $L_1([0,1])$. We need a particular way of partitioning $[0,1]$, using elements of $\mathbb{T}(3)$. Define $E_\emptyset = [0,1]$, recalling that $\emptyset = () \in \mathbb{T}(3)$. Once we have partitioned $[0,1]$ into 3^n measurable subsets each of measure 3^{-n}, $\{E_\tau : \tau \in \mathbf{3}^n\}$, we obtain the next level of partitioning by partitioning E_τ into three measurable subsets of equal measure and label them $E_{\tau\oplus(0)}$, $E_{\tau\oplus(1)}$ and $E_{\tau\oplus(2)}$.

The Individual Operator Problem

We define a system of functions, s_τ indexed by elements of $\mathbb{T}(\mathbb{N}^*)$ and write $\mathbb{S}(s_\tau)$ for the support, specified up to a set of measure zero, of s_τ. Start by defining $s_\emptyset$ to be the zero function on $[0,1]$. Suppose that $\tau \in \mathbb{T}(\mathbb{N}^*)$ with length $n-1$ and s_τ has been defined. If $k \in \mathbb{N}^*$, define

$$G_{\tau\oplus(k)} = \begin{cases} E_{\Phi_n(k)} & \text{if } E_{\Phi_n(k)} \text{ is disjoint from } \mathbb{S}(s_\tau) \\ \emptyset & \text{otherwise.} \end{cases}$$

It is important that this definition depends, for a given τ, only on $k \mod 3^n$. Now set

$$s_{\tau\oplus(k)} = s_\tau + \boldsymbol{r}_k \mathbf{1}_{G_{\tau\oplus(k)}},$$

where $\boldsymbol{r}_k$ is the k'th Rademacher function.

The Individual Operator Problem

Example (Elliott, 2014)

The operator $S \in \mathcal{L}^r\big(L_1([0,1]), F\big)$, defined by the formula

$$Sf = \big(s_\tau(f)\big)_{\tau \in \mathbb{T}(\mathbb{N}^*)},$$

where $s_\tau(f) = \int_0^1 s_\tau(t) f(t)\, dt$, has a modulus which is not given by the Riesz-Kantorovich formula.