Mixing inequalities in Riesz spaces[2]

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- Markov processes, are defined in terms of independence.
- Mixingales[4] are processes which exhibit independence/conditional independence in the limit.
- Mixing processes are dependent stochastic processes in which measures of independence (the so called mixing coefficients) are use to deduce structure.[5] [6]
- In this talk we will be concerned with two such mixing coefficients: the strong or α mixing coefficient and the uniform or φ mixing coefficient.

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4 W.-C. Kuo, J.J. Vardy, B.A. Watson, Mixingales on Riesz spaces, *J. Math. Anal. Appl.*, **402** (2013), 731-738.

5 P.P. Billingsley, *Probability and Measure*, John Wiley and Sons, 3rd edition, 1995.

6 P. Doukhan, Mixing: properties and examples, *Lecture Notes in Statistics*, **85** (1994), 15-23.

The strong or α mixing coefficient in a probability space

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{F}$.

- The strong mixing coefficient between $\mathcal{A}$ and $\mathcal{B}$ is

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mu(A \cap B) - \mu(A)\mu(B)| \,|\, A \in \mathcal{A}, B \in \mathcal{B}\}. \quad (1)$$

- The observation that $\mu(A) = \mathbb{E}[\mathbb{I}_A|\{\phi, \Omega\}]$ leads one the following definition for a conditional strong mixing coefficient. If $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{A} \cap \mathcal{B}$ then the $\mathcal{C}$-conditioned strong mixing coefficient of $\mathcal{A}$ and $\mathcal{B}$ is

$$\alpha_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{E}[\mathbb{I}_A\mathbb{I}_B|\mathcal{C}] - \mathbb{E}[\mathbb{I}_A|\mathcal{C}]\mathbb{E}[\mathbb{I}_B|\mathcal{C}] \,|\, A \in \mathcal{A}, B \in \mathcal{B}]. \quad (2)$$

The uniform or φ mixing coefficient in a probability space

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{F}$.

- The uniform mixing coefficient between $\mathcal{A}$ and $\mathcal{B}$ is

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup\{|\mu(B|A) - \mu(B)| \, | \, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0\}. \quad (3)$$

- The uniform mixing coefficient between $\mathcal{A}$ and $\mathcal{B}$ is

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup\{|\mu(B|A) - \mu(B)| \, | \, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0\}. \quad (4)$$

Lemma

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{F}$, then

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup_{B \in \mathcal{B}} \|\mathbb{E}[\mathbb{I}_B - \mathbb{E}[\mathbb{I}_B]|\mathcal{A}]\|_\infty .$$

The mixing coefficients

- For both the strong and uniform mixing coefficients we have that the coefficient is zero if and only if the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ are independent.
- Both mixing coefficients lie in the interval $[0, 1]$.

- In the probability space $(\Omega, \mathcal{F}, \mu)$ if f is $\mathcal{F}$ measurable then for $1 \leq p \leq r \leq \infty$ we have

$$\begin{aligned}\|\mathbb{E}[f|\mathcal{B}] - \mathbb{E}[f]\|_p &\leq 2[\varphi(\mathcal{B},\mathcal{F})]^{1-1/r}\|f\|_r,\\ \|\mathbb{E}[f|\mathcal{B}] - \mathbb{E}[f]\|_p &\leq (2^{1/p}+1)[\alpha(\mathcal{B},\mathcal{F})]^{1/p-1/r}\|f\|_r.\end{aligned}$$

- First proved by Ibragimov[11]
- Using these inequalities McLeisch[12] showed that mixing processes (with some extra conditions) asymptotically approach a Brownian motion.
- Serfling[13] used these inequalities to give a central limit theorem for mixing processes.

[11] I.A. IBRAGIMOV, Some limit theorems for stationary processes, *Theory Probab. Appl.*, **7** (1962), 349-382.

[12] D.L. MCLEISH, A maximal inequality and dependent strong laws, *Ann. Probab.*, **3** (1975), 829-839.

[13] R.J. SERFLING, Contributions to central limit theory for dependent variables, *Ann. Math. Stat.*, **39** (1968), 1158-1175.

Conditional expectation operators in Riesz spaces

Let E be a Dedekind complete Riesz space with weak order unit. If

- T is a positive order continuous linear projection T on E
- $R(T)$ is a Dedekind complete Riesz subspace of E
- Te is a weak order unit of E for each weak order unit e of E

then T is said to be a conditional expectation operator on E.

Maximal extensions

- We say that a conditional expectation operator, T, on a Riesz space is strictly positive if $T|f| = 0$ implies that $f = 0$.
- It was shown[16] that a strictly positive conditional expectation operator, T, on a Riesz space, E, admits a unique maximal extension to a conditional expectation operator, also denoted T, in the universal completion, E^u, of E, with domain a Dedekind complete Riesz space, which will be denoted $L^1(T)$.
- The procedure used there was based on that of de Pagter and Grobler, [17], for the measure theoretic setting.
- If $(\Omega, \mathcal{F}, \mu)$ is a probability space and Tf if the a.e. constant function $\int_\Omega f\, d\mu$ then $L^1(T) = L^1(\Omega, \mathcal{F}, \mu)$ if $\mathbb{I}_A \in E$ for all $A \in \mathcal{F}$.

[16] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Conditional expectations on Riesz spaces, *J. Math. Anal. Appl.*, **303** (2005), 509-521.

[17] J.J. Grobler, B. de Pagter, Operators representable as multiplication-conditional expectation operators, *J. Operator Theory*, **48** (2002), 15-40.

$L^1(T)$ as an $R(T)$-module

- Let $R(T)$ denote the range of the maximal extension of the conditional expectation operator, i.e. $R(T) := \{Tf \,|\, f \in L^1(T)\}$.
- $R(T)$ is a universally complete f-algebra. For the measure version see [19].
- $L^1(T)$ is an $R(T)$-module. For the measure version see [20].
- This prompts the definition of an $R(T)$ (vector valued) norm $\|\cdot\|_{T,1} := T|\cdot|$ on $L^1(T)$. Here the homogeneity is with respect to multiplication by elements of $R(T)_+$.

[19] J.J. GROBLER, B. DE PAGTER, Operators representable as multiplication-conditional expectation operators, *J. Operator Theory*, **48** (2002), 15-40.

[20] S. CERREIA-VIOGLIO, M. KUPPER, F. MACCHERONI, M. MARINACCI, N. VOGELPOTH, Conditional L^p-spaces and the duality of modules over f-algebras, *J. Math. Anal. Appl.*, (2016), in press

Definition

Let E be a Dedekind complete Riesz space with weak order unit and T be a strictly positive conditional expectation operator on E. If E is an $R(T)$-module and $\phi : E \to R(T)_+$ with

(a) $\phi(f) = 0$ if and only if $f = 0$,

(b) $\phi(gf) = |g|\phi(f)$ for all $f \in E$ and $g \in R(T)$,

(c) $\phi(f + h) \leq \phi(f) + \phi(h)$ for all $f, h \in E$,

then ϕ will be called an $R(T)$-valued norm on E.

$L^\infty(T)$

- We take $L^\infty(T)$ to be the subspace of $L^1(T)$ composed of $R(T)$ bounded elements, i.e.

$$L^\infty(T) := \{f \in L^1(T) \mid |f| \le g, \text{ for some } g \in R(T)_+\}.$$

- The map

$$f \mapsto \|f\|_{T,\infty} := \inf\{g \in R(T)_+ \mid |f| \le g\},$$

  for $f \in L^\infty(T)$ defines an $R(T)$ valued norm on $L^\infty(T)$.
- This extends on the concepts of $L^\infty(T)$ defined in [23].
- T is an averaging operator in the sense that if $f \in R(T)$ and $g \in E$ with $fg \in E$ then $T(fg) = fT(g)$.

[23] C.C.A. LABUSCHAGNE, B.A. WATSON, Discrete stochastic integration in Riesz spaces, *Positivity*, **14** (2010), 859-875.

Some inequalities

Theorem (Hölders's Inequality)

If $f \in L^1(T)$ and $g \in L^\infty(T)$, then $gf \in L^1(T)$ and

$$\|gf\|_{T,1} \leq \|g\|_{T,\infty}\|f\|_{T,1}.$$

Theorem (Jensen's Inequality)

If S is a conditional expectation operator on $L^1(T)$ compatible T (in the sense that $TS = T = ST$), then

$$\|Sf\|_{T,p} \leq \|f\|_{T,p},$$

for all $f \in L^p(T), p = 1, \infty$.

Mixing coefficients in Riesz spaces

Let E be a Dedekind complete Riesz space with weak order unit, say e, and conditional expectation operator, T, with $Te = e$.

- If U is a conditional expectation operators on E, with $TU = T = UT$, then we say that U is compatible with T.
- If U is a conditional expectation on E compatible with T then we denote by $\mathcal{B}(U)$ the set of band projections P on E with $Pe \in R(U)$.
- We define the T-conditioned strong mixing coefficient with respect to the conditional expectation operators U and V on E compatible with T, by

$$\alpha_T(U, V) := \sup\{|TPQe - TPe \cdot TQe| \mid P \in \mathcal{B}(U), Q \in \mathcal{B}(V)\}.$$

- Let U and V be conditional expectation operators on E compatible with T, then

$$\varphi_T(U, V) = \sup_{Q \in \mathcal{B}(V)} \|UQe - TQe\|_{T,\infty}.$$

Mixing inequalities in Riesz spaces

- Let E be a T-universally complete Riesz space, $E = L^1(T)$, where T is a conditional expectation operator on E where E has a weak order unit, say e, with $Te = e$.
- Let U and V be conditional expectation operators on E compatible with T.
- Then for $f \in R(V) \cap L^\infty(T)$, we have

$$\begin{aligned} \|Uf - Tf\|_{T,1} &\leq 4\alpha_T(U,V)\|f\|_{T,\infty}, \\ \|Uf - Tf\|_{T,1} &\leq \|Uf - Tf\|_{T,\infty} \leq 2\varphi_T(U,V)\|f\|_{T,\infty}. \end{aligned}$$

- For proofs see JMAA online first or Arxiv ....
- Using the mixing inequalities above, Riesz space mixing processes can be connected to Riesz space mixingales[27] and thus obey a law of large numbers.

[27] W.-C. Kuo, J.J. Vardy, B.A. Watson, Mixingales on Riesz spaces, *J. Math. Anal. Appl.*, **402** (2013), 731-738.

- A consideration of σ-finite processes in the context of martingale theory can be found in the work of Dellacherie and Meyer, [29].
- Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, which to be interesting should have $\mu(\Omega) = \infty$, and let $(\Omega_i)_{i\in\mathbb{N}}$ be a μ-measurable partition of Ω into sets of finite positive measure.
- Let $\mathcal{A}_0$ be the sub-σ-algebra of $\mathcal{A}$ generated by $(\Omega_i)_{i\in\mathbb{N}}$. We take the Riesz space $E = L^\infty(\Omega, \mathcal{A}, \mu)$ and the conditional expectation operator $T = \mathbb{E}[\,\cdot\,|\mathcal{A}_0]$.
- For $f \in E$ we have

$$Tf(\omega) = \frac{\int_{\Omega_i} f\,d\mu}{\mu(\Omega_i)}, \quad \text{for} \quad \omega \in \Omega_i. \tag{5}$$

[29] Sections 39, 42 and 43 of C. DELLACHERIE, P.-A. MEYER, *Probabilities and Potentials: B, Theory of Martingales,* North Holland Publishing Company, 1982.

Application to σ-finite processes - spaces

- The universal completion, E^u, of E is the space of all $\mathcal{A}$-measurable functions.
- The T-universal completion of E is the space

$$\mathcal{L}^1(T) = \left\{ f \in E^u \;\middle|\; \int_{\Omega_i} |f| \, d\mu < \infty \text{ for all } i \in \mathbb{N} \right\},$$

  which is characterized by $f|_{\Omega_i} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $i \in \mathbb{N}$.
- Here T can be extended to an $\mathcal{L}^1(T)$ conditional expectation operator as per (5).
- The space E has a weak order unit $e = 1$, the function identically 1 on Ω, which again is a weak order unit for $\mathcal{L}^1(T)$, but is not in $L^1(\Omega, \mathcal{A}, \mu)$.
- The range of the generalized conditional expectation operator T is

$$R(T) = \{ f \in E^u \mid f \text{ a.e. constant on } \Omega_i, i \in \mathbb{N} \},$$

  which is an f-algebra.

Application to σ-finite processes - vector norms

- The last of the spaces to be considered is

$$\mathcal{L}^\infty(T) = \{f \in E^u \mid f \text{ essentially bounded on } \Omega_i \text{ for each } i \in \mathbb{N}\}.$$

- The vector norms on $\mathcal{L}^1(T)$ and $\mathcal{L}^\infty(T)$ are

$$\|f\|_{T,1}(\omega) = T|f|(\omega) = \frac{\int_{\Omega_i} |f|\, d\mu}{\mu(\Omega_i)}, \quad \text{for} \quad \omega \in \Omega_i, f \in \mathcal{L}^1(T), \tag{6}$$

$$\|f\|_{T,\infty}(\omega) = \text{ess sup}_{\Omega_i} |f|, \quad \text{for} \quad \omega \in \Omega_i, f \in \mathcal{L}^\infty(T). \tag{7}$$

- Note that $L^1(\Omega, \mathcal{A}, \mu) \subsetneq \mathcal{L}^1(T)$, $L^\infty(\Omega, \mathcal{A}, \mu) \subsetneq \mathcal{L}^\infty(T)$, $\mathcal{L}^\infty(T) \subset \mathcal{L}^1(T)$ while $L^\infty(\Omega, \mathcal{A}, \mu) \not\subset L^1(\Omega, \mathcal{A}, \mu)$.

Application to σ-finite processes - conditioning

- Let $\mathcal{C}$ and $\mathcal{D}$ be sub-σ-algebras of $\mathcal{A}$ which contain $\mathcal{A}_0$.
- The α-mixing coefficient of $\mathcal{C}$ and $\mathcal{D}$ conditioned on $\mathcal{A}_0$ (which in measure theoretic terms could be denote $\alpha_{\mathcal{A}_0}(\mathcal{C},\mathcal{D})$ is $\alpha_T(U,V)$.
- Here U and V are the restrictions to $\mathcal{L}^1(T)$ of the extensions to $\mathcal{L}^1(U)$ and $\mathcal{L}^1(V)$ respectively of the conditional expectation operators U and V on E conditioning with respect to the σ-algebras $\mathcal{C}$ and $\mathcal{D}$.
- Explicitly

$$U(f) = \sum_{i=1}^{\infty} \mathbb{E}_i[f\mathbb{I}_{\Omega_i}|\mathcal{C}], \tag{8}$$

$$V(f) = \sum_{i=1}^{\infty} \mathbb{E}_i[f\mathbb{I}_{\Omega_i}|\mathcal{D}], \tag{9}$$

for $f \in \mathcal{L}^1(T)$.

Application to σ-finite processes - α

- The conditional expectation $\mathbb{E}_i[f\mathbb{I}_{\Omega_i}|\mathcal{C}] = \mathbb{E}_i[f|\mathcal{C}]$ is the conditional expectation on Ω_i of $f|_{\Omega_i}$ with respect to the probability measure $\mu_i(A) := \frac{\mu(A\cap\Omega_i)}{\mu(\Omega_i)}$ and the σ-algebra $\{C \cap \Omega_i | C \in \mathcal{C}\}$, and similarly for $\mathcal{C}$ replaced by $\mathcal{D}$.
- Explicitly

$$\alpha_T(U,V) = \alpha_{\mathcal{A}_0}(\mathcal{C},\mathcal{D}) = \sum_{i=1}^{\infty} \alpha_i(\mathcal{C},\mathcal{D})\mathbb{I}_{\Omega_i},$$

where $\alpha_i(\mathcal{C},\mathcal{D})$ is the α-mixing coefficient of σ-algebras $\mathcal{C}$ and $\mathcal{D}$ with respect to the probability measure μ_i.

Application to σ-finite processes - α inequality

- For g is μ-measurable and essential bounded on each $\Omega_i, i \in \mathbb{N}$, we have

$$\|UVg - Tg\|_{T,1} \leq 4\alpha_T(U,V)\|g\|_{T,\infty},$$

which in this example case can be written as, for each $i \in \mathbb{N}$,

$$\frac{1}{\mu(\Omega_i)} \int_{\Omega_i} \left| \mathbb{E}_i[\mathbb{E}_i[g|\mathcal{D}]|\mathcal{C}] - \frac{1}{\mu(\Omega_i)} \int_{\Omega_i} g \, d\mu \right| d\mu \leq 4\alpha_i(\mathcal{C},\mathcal{D})\text{ess sup}_{\Omega_i}|g|$$

Application to σ-finite processes - φ

- The conditional uniform mixing coefficient is given by

$$\varphi_T(U,V) = \varphi_{\mathcal{A}_0}(\mathcal{C},\mathcal{D}) = \sum_{i=1}^{\infty} \varphi_i(\mathcal{C},\mathcal{D})\mathbb{I}_{\Omega_i},$$

where $\varphi_i(\mathcal{C},\mathcal{D})$ is the φ-mixing coefficient of $\mathcal{C}$ and $\mathcal{D}$ relative to the probability measure μ_i.

- For g is μ-measurable and essential bounded on each $\Omega_i, i \in \mathbb{N}$, we have

$$\|UVg - Tg\|_{T,\infty} \leq 2\varphi_T(U,V)\|g\|_{T,\infty}.$$

- Explicitly

$$\left|\mathbb{E}_i[\mathbb{E}_i[g|\mathcal{D}]|\mathcal{C}] - \frac{1}{\mu(\Omega_i)}\int_{\Omega_i} g\, d\mu\right| \leq 2\varphi_i(\mathcal{C},\mathcal{D}) \text{ ess sup }_{\Omega_i}|g|.$$

THANK YOU