Amenability of locally compact quantum groups and their unitary co-representations

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Amenability of groups

G – a locally compact group.

Definition

A mean on G is a state $m \in L^\infty(G)^*$
(that is: $m(x) \geq 0$ when $x \geq 0$ and $m(\mathbb{1}) = 1$).

A mean m is left invariant if $m(L_t x) = m(x)$ for all $x \in L^\infty(G)$ and $t \in G$.

G is amenable if it has a left invariant mean.

Examples (and non-examples)

1. Every compact group is amenable: use the Haar measure!
2. Every abelian (or even solvable) group is amenable
   - Markov–Kakutani fixed point theorem
3. Every locally-finite group is amenable
4. $\mathbb{F}_n$ is *not* amenable for all $n \geq 2$.

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Amenability of groups

Numerous equivalent characterizations:

G is amenable $\iff$ ...

- topological amenability: there is a mean $m \in L^\infty(G)^*$ with

$$m(\omega * x) = \omega(\mathbb{1})m(x) \qquad \text{for all } x \in L^\infty(G), \omega \in L^1(G)$$

- Leptin's theorem: $\mathrm{VN}(G)_*$ has a left bounded approximate identity
  - here $\mathrm{VN}(G) := \langle \lambda_g : g \in G \rangle \subseteq B(L^2(G))$ and $\mathrm{VN}(G)_* \cong A(G)$.

[But also: means on algebras other than $L^\infty(G)$, Hulanicki's theorem, Folner's condition, Reiter's condition(s), Rickert's theorem, etc...]

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Amenability of groups

Theorem (Lance, '73)

1. *G is amenable $\implies$ $C_r^*(G)$ is nuclear*
2. *if G is discrete, the converse also holds*
   - *(but not generally).*

The reduced group C^*-algebra $C_r^*(G)$

Recall: $(\lambda_g)_G$ is the left regular rep of G on $L^2(G)$.

$$C_r^*(G) := \overline{\left\{\int_G f(t)\lambda_t \, \mathrm{d}t : f \in C_c(G)\right\}}^{\|\cdot\|} \subseteq \langle \lambda_g : g \in G \rangle =: \mathrm{VN}(G).$$

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Nuclearity

Definition

A C^*-algebra A is nuclear if

"the identity map A $\to$ A approximately factors through fin-dim algebras via CP contractions",

that is: there are nets (n_α) in $\mathbb{N}$ and $(S_\alpha), (T_\alpha)$ of completely positive contractions

$$\begin{tikzcd} A \arrow[dr, "S_\alpha"'] & & A \\ & M_{n_\alpha}(\mathbb{C}) \arrow[ur, "T_\alpha"'] & \end{tikzcd}$$

and $(T_\alpha \circ S_\alpha)(x) \underset{\alpha}{\longrightarrow} x$ for every $x \in A$.

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Remark

A is nuclear $\iff$ A^{**} is injective $\iff$ every representation of A generates an injective vN alg.

Recall: by Lance, G is amenable $\implies$ $C_r^*(G)$ is nuclear, but the converse does not always hold.

Question

Find a characterization of amenability involving nuclearity that always works.

Several similar characterizations involving injectivity were found recently (Soltan–V, Crann–Neufang, Crann).

Theorem (C.-K. Ng, '15)

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A group as a quantum group

G – a locally compact group.

1. The von Neumann algebra $L^\infty(G)$.
2. Co-multiplication: the $*$-homomorphism

$$\Delta : L^\infty(G) \to L^\infty(G) \,\overline{\otimes}\, L^\infty(G) \cong L^\infty(G \times G)$$

   defined by

$$(\Delta(f))(t,s) := f(ts) \qquad (f \in L^\infty(G)).$$

   By associativity, we have $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

3. Left and right Haar measures. View them as n.s.f. weights $\varphi, \psi : L^\infty(G)_+ \to [0,\infty]$ by $\varphi(f) := \int_G f(t)\,\mathrm{d}t_\ell$, $\psi(f) := \int_G f(t)\,\mathrm{d}t_r$.

Motivation for quantum groups

Lack of Pontryagin duality for non-Abelian l.c. groups.

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Locally compact quantum groups

Definition (Kustermans–Vaes, '00)

A locally compact quantum group is a pair $\mathbb{G} = (M, \Delta)$ such that:

1. M is a von Neumann algebra
2. $\Delta : M \to M \overline{\otimes} M$ is a co-multiplication: a normal, faithful, unital $*$-homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

3. There are two n.s.f. weights φ, ψ on M (the Haar weights) with:
   - $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
   - $\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\psi(x) < \infty$.

Denote $L^\infty(\mathbb{G}) := M$.

$L^\infty(\mathbb{G})_*$ becomes a Banach algebra by $\omega * \rho := (\omega \otimes \rho) \circ \Delta$.

$C_0(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$: canonical weakly dense C^*-algebra.

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Locally compact quantum groups

Rich structure theory, including an unbounded antipode and duality $\mathbb{G} \mapsto \hat{\mathbb{G}}$ within the category satisfying $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

Example (commutative LCQGs: $\mathbb{G} = G$)

$L^\infty(\mathbb{G}) = L^\infty(G)$
$C_0(\mathbb{G}) = C_0(G)$

Example (co-commutative LCQGs: $\mathbb{G} = \hat{G}$)

The dual $\hat{G}$ of G (as a LCQG) has

- $L^\infty(\mathbb{G}) = \mathrm{VN}(G)$
  $C_0(\mathbb{G}) = C_r^*(G)$
- $\Delta : \mathrm{VN}(G) \to \mathrm{VN}(G) \overline{\otimes} \mathrm{VN}(G)$ given by $\Delta(\lambda_g) := \lambda_g \otimes \lambda_g$
- $\varphi = \psi =$ the Plancherel weight on $\mathrm{VN}(G)$.

If G is Abelian, $\hat{G}$ is its Pontryagin dual (up to unitary equivalence).

Amenability of LCQGs

Recall: a group G is amenable $\iff$ there is a mean $m \in L^\infty(G)^*$ with $m(\omega * x) = \omega(\mathbb{1})m(x)$ for all $x \in L^\infty(G)$ and $\omega \in L^1(G) \cong L^\infty(G)_*$.

Definition

A LCQG $\mathbb{G}$ is amenable if there is a state $m \in L^\infty(\mathbb{G})^*$ such that

$$m\Big(\underbrace{(\mathrm{id} \otimes \omega)(\Delta(x))}_{\omega * x}\Big) = \omega(\mathbb{1})m(x) \qquad (\forall x \in L^\infty(\mathbb{G}), \omega \in L^\infty(\mathbb{G})_*).$$

Recall Leptin's theorem: a group G is amenable $\iff$ $\mathrm{VN}(G)_*$ has a left bounded approximate identity.

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At least one side of Leptin's theorem is true:

Theorem (Bédos–Tuset, '03)

If $\mathbb{G}$ is strongly amenable, then it is amenable.

The converse is open even for Kac algebras.

The left regular representation

$\mathbb{G}$ – locally compact quantum group.
All information on $\mathbb{G}$ is encoded in a unitary $W \in \mathrm{M}(C_0(\mathbb{G}) \otimes_{\min} C_0(\hat{\mathbb{G}}))$ satisfying

$$\Delta(x) = W^*(\mathbb{1} \otimes x)W \qquad (\forall x \in L^\infty(\mathbb{G})).$$

Example

If $\mathbb{G} = G$, then $W \in \mathrm{M}(C_0(G) \otimes_{\min} C_r^*(G)) \cong C_b(G, \mathrm{M}(C_r^*(G)))$ given by

$$g \mapsto \lambda_g \qquad (g \in G).$$

Characterizing amenability in terms of nuclearity

Recall Ng's Theorem:

Theorem

G – locally compact group.
G is amenable $\iff$ $C_r^*(G)$ is nuclear and has a tracial state.

Main Theorem 1 (Ng–V., '17)

$\mathbb{G}$ – LCQG. Consider the following conditions:

1. $\mathbb{G}$ is strongly amenable;
2. $C_0(\hat{\mathbb{G}})$ is nuclear, and there is a state ρ on $C_0(\hat{\mathbb{G}})$ that is invariant under the left action of $\mathbb{G}$ on $C_0(\hat{\mathbb{G}})$:
$$(\mathrm{id} \otimes \rho)\left(W^*(\mathbb{1} \otimes x)W\right) = \rho(x)\mathbb{1} \qquad (\forall x \in C_0(\hat{\mathbb{G}}));$$
3. $\mathbb{G}$ is amenable.

Then $1 \implies 2 \implies 3$.

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Invariance under the left action of $\mathbb{G}$ on $C_0(\hat{\mathbb{G}})$

Case $\mathbb{G} = G$

W corresponds to the function $g \mapsto \lambda_g$,
thus, for $x \in C_r^*(G)$, $W^*(\mathbb{1} \otimes x)W$ corresponds to the func. $g \mapsto \lambda_g^* x \lambda_g$,
hence the condition

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means that $\rho \in C_r^*(G)^*$ satisfies

$$\rho(\lambda_g^* x \lambda_g) = \rho(x) \qquad (\forall g \in G),$$

that is, ρ is tracial.

Case $\mathbb{G}$ is discrete

The Haar state of $\hat{\mathbb{G}}$ satisfies the invariance condition $\iff$ $\mathbb{G}$ is Kac (Izumi, '02).

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3. $\mathbb{G}$ is amenable.

Then $1 \Longrightarrow 2 \Longrightarrow 3$.

Main Theorem 2 (Crann, '17, yet unpublished)

We have $1 \iff 2$.

We emphasize that his proof uses our implication $2 \Longrightarrow 3$ to deduce 1 from 2.

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$\mathbb{G}$ – LCQG. Consider the following conditions:

1. $\mathbb{G}$ is strongly amenable;
2. $C_0(\hat{\mathbb{G}})$ is nuclear, and there is a state ρ on $C_0(\hat{\mathbb{G}})$ that is invariant under the left action of $\mathbb{G}$ on $C_0(\hat{\mathbb{G}})$;
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Then $1 \implies 2 \implies 3$.

Main Theorem 2 (Crann, '17, yet unpublished)

We have $1 \iff 2$.

We emphasize that his proof uses our implication $2 \implies 3$ to deduce 1 from 2.

Characterizing amenability in terms of nuclearity

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Tools of proof of Main Theorem 1

- Implication 1 $\implies$ 2 is easy.
- For 2 $\implies$ 3, consider the following:

Definition and Theorem (Bekka, '90, Inventiones)

G – locally compact group.

Def. A representation π of G on $\mathcal{H}$ is amenable if there exists a G-invariant mean on $B(\mathcal{H})$: a state $m \in B(\mathcal{H})^*$ such that $m\left(\pi_g^* x \pi_g\right) = m(x)$ for all $x \in B(\mathcal{H})$, $g \in G$.

Thm. If $\pi_1 \leq \pi_2$ and π_1 is amenable, so is π_2.

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Bédos, Conti and Tuset ('03 and '05) extended Bekka's definition for LCQG representations. They asked:

Question

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They were able to give partial answers.

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The Bédos–Conti–Tuset question has an affirmative answer.

Proof of Main Theorem 1, implication 2 $\implies$ 3

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Thank you for your attention!