

Amenability of locally compact quantum groups and their unitary co-representations

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Amenability of groups

G – a locally compact group.

Definition

A **mean** on G is a state $m \in L^\infty(G)^*$

(that is: $m(x) \geq 0$ when $x \geq 0$ and $m(\mathbf{1}) = 1$).

A mean m is **left invariant** if $m(L_t x) = m(x)$ for all $x \in L^\infty(G)$ and $t \in G$.

G is **amenable** if it has a left invariant mean.

Examples (and non-examples)

- 1 Every **compact** group is amenable: use the Haar measure!
- 2 Every **abelian** (or even **solvable**) group is amenable
 - ▶ Markov–Kakutani fixed point theorem
- 3 Every **locally-finite** group is amenable
- 4 \mathbb{F}_n is *not* amenable for all $n \geq 2$.

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Numerous equivalent characterizations:

G is amenable \iff ...

- **topological amenability**: there is a mean $m \in L^\infty(G)^*$ with

$$m(\omega * x) = \omega(\mathbf{1})m(x) \quad \text{for all } x \in L^\infty(G), \omega \in L^1(G)$$

- **Leptin's theorem**: $VN(G)_*$ has a left bounded approximate identity
 - here $VN(G) := \langle \lambda_g : g \in G \rangle \subseteq B(L^2(G))$ and $VN(G)_* \cong A(G)$.

[But also: means on algebras other than $L^\infty(G)$, Hulanicki's theorem, Folner's condition, Reiter's condition(s), Rickert's theorem, etc...]

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Theorem (Lance, '73)

- 1 G is *amenable* $\implies C_r^*(G)$ is *nuclear*
- 2 if G is discrete, the converse also holds
 - ▶ (but not generally).

The reduced group C^* -algebra $C_r^*(G)$

Recall: $(\lambda_g)_G$ is the left regular rep of G on $L^2(G)$.

$$C_r^*(G) := \overline{\left\{ \int_G f(t)\lambda_t dt : f \in C_c(G) \right\}}^{\|\cdot\|} \subseteq \langle \lambda_g : g \in G \rangle =: VN(G).$$

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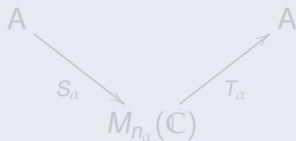
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A C^* -algebra A is **nuclear** if

“the identity map $A \rightarrow A$ approximately factors through fin-dim algebras via CP contractions”,

that is: there are nets (n_α) in \mathbb{N} and $(S_\alpha), (T_\alpha)$ of completely positive contractions



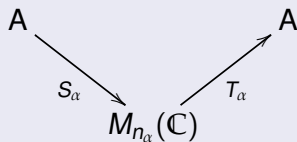
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Nuclearity

Remark

A is **nuclear** \iff A^{**} is **injective** \iff every representation of A generates an **injective** vN alg.

Recall: by Lance, G is **amenable** $\implies C_r^*(G)$ is **nuclear**, but the converse does not always hold.

Question

Find a characterization of **amenability** involving **nuclearity** that always works.

Several similar characterizations involving injectivity were found recently (Soltan–V, Crann–Neufang, Crann).

Theorem (C.-K. Ng, '15)

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A group as a quantum group

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- 1 The von Neumann algebra $L^\infty(G)$.
- 2 **Co-multiplication**: the $*$ -homomorphism

$$\Delta : L^\infty(G) \rightarrow L^\infty(G) \bar{\otimes} L^\infty(G) \cong L^\infty(G \times G)$$

defined by

$$(\Delta(f))(t, s) := f(ts) \quad (f \in L^\infty(G)).$$

By associativity, we have $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$.

- 3 **Left and right Haar measures**. View them as n.s.f. weights $\varphi, \psi : L^\infty(G)_+ \rightarrow [0, \infty]$ by $\varphi(f) := \int_G f(t) dt_\ell$, $\psi(f) := \int_G f(t) dt_r$.

Motivation for quantum groups

Lack of Pontryagin duality for non-Abelian l.c. groups.

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Definition (Kustermans–Vaes, '00)

A **locally compact quantum group** is a pair $\mathbb{G} = (M, \Delta)$ such that:

- 1 M is a **von Neumann algebra**
- 2 $\Delta : M \rightarrow M \bar{\otimes} M$ is a **co-multiplication**: a normal, faithful, unital $*$ -homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

- 3 There are two n.s.f. weights φ, ψ on M (the **Haar weights**) with:
 - ▶ $\varphi((\omega \otimes \text{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
 - ▶ $\psi((\text{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\psi(x) < \infty$.

Denote $L^\infty(\mathbb{G}) := M$.

$L^\infty(\mathbb{G})_*$ becomes a Banach algebra by $\omega * \rho := (\omega \otimes \rho) \circ \Delta$.

$C_0(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$: canonical weakly dense C^* -algebra.

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Locally compact quantum groups

Rich structure theory, including an unbounded **antipode** and **duality**
 $G \mapsto \hat{G}$ within the category satisfying $\hat{\hat{G}} = G$.

Example (commutative LCQGs: $G = G$)

$$L^\infty(\mathbb{G}) = L^\infty(G)$$

$$C_0(\mathbb{G}) = C_0(G)$$

Example (co-commutative LCQGs: $G = \hat{G}$)

The dual \hat{G} of G (as a LCQG) has

- $L^\infty(\mathbb{G}) = \text{VN}(G)$
 $C_0(\mathbb{G}) = C_r^*(G)$
- $\Delta : \text{VN}(G) \rightarrow \text{VN}(G) \bar{\otimes} \text{VN}(G)$ given by $\Delta(\lambda_g) := \lambda_g \otimes \lambda_g$
- $\varphi = \psi =$ the Plancherel weight on $\text{VN}(G)$.

If G is Abelian, \hat{G} is its Pontryagin dual (up to unitary equivalence).

Amenability of LCQGs

Recall: a group G is **amenable** \iff there is a mean $m \in L^\infty(G)^*$ with $m(\omega * x) = \omega(\mathbb{1})m(x)$ for all $x \in L^\infty(G)$ and $\omega \in L^1(G) \cong L^\infty(G)_*$.

Definition

A LCQG \mathbb{G} is **amenable** if there is a state $m \in L^\infty(\mathbb{G})^*$ such that

$$m\left(\underbrace{(\text{id} \otimes \omega)(\Delta(x))}_{\omega * x}\right) = \omega(\mathbb{1})m(x) \quad (\forall x \in L^\infty(\mathbb{G}), \omega \in L^\infty(\mathbb{G})_*).$$

Recall Leptin's theorem: a group G is **amenable** \iff $VN(G)_*$ has a left bounded approximate identity.

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At least one side of Leptin's theorem is true:

Theorem (Bédos–Tuset, '03)

If G is *strongly amenable*, then it is *amenable*.

The converse is open even for Kac algebras.

The left regular representation

\mathbb{G} – locally compact quantum group.

All information on \mathbb{G} is encoded in a unitary $W \in M(C_0(\mathbb{G}) \otimes_{\min} C_0(\hat{\mathbb{G}}))$ satisfying

$$\Delta(x) = W^*(\mathbb{1} \otimes x)W \quad (\forall x \in L^\infty(\mathbb{G})).$$

Example

If $\mathbb{G} = G$, then $W \in M(C_0(G) \otimes_{\min} C_r^*(G)) \cong C_b(G, M(C_r^*(G)))$ given by

$$g \mapsto \lambda_g \quad (g \in G).$$

Characterizing amenability in terms of nuclearity

Recall Ng's Theorem:

Theorem

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G is **amenable** $\iff C_r^*(G)$ is **nuclear** and has a **tracial state**.

Main Theorem 1 (Ng–V., '17)

G – LCQG. Consider the following conditions:

- 1 G is **strongly amenable**;
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Then $1 \implies 2 \implies 3$.

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Invariance under the left action of \mathbb{G} on $C_0(\hat{\mathbb{G}})$

Case $\mathbb{G} = G$

W corresponds to the function $g \mapsto \lambda_g$,

thus, for $x \in C_r^*(G)$, $W^*(\mathbb{1} \otimes x)W$ corresponds to the func. $g \mapsto \lambda_g^* x \lambda_g$,
hence the condition

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means that $\rho \in C_r^*(G)^*$ satisfies

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that is, ρ is tracial.

Case \mathbb{G} is discrete

The Haar state of $\hat{\mathbb{G}}$ satisfies the invariance condition $\iff \mathbb{G}$ is Kac (Izumi, '02).

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- 3 G is **amenable**.

Then $1 \implies 2 \implies 3$.

Main Theorem 2 (Crann, '17, yet unpublished)

We have $1 \iff 2$.

We emphasize that his proof uses our implication $2 \implies 3$ to deduce 1 from 2.

Characterizing amenability in terms of nuclearity

Recall our result:

Main Theorem 1 (Ng–V.)

G – LCQG. Consider the following conditions:

- 1 G is **strongly amenable**;
- 2 $C_0(\hat{G})$ is **nuclear**, and there is a state ρ on $C_0(\hat{G})$ that is **invariant under the left action of G** on $C_0(\hat{G})$;
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Then $1 \implies 2 \implies 3$.

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G – locally compact group.

Def. A representation π of G on \mathcal{H} is **amenable** if there exists a **G -invariant mean** on $B(\mathcal{H})$: a state $m \in B(\mathcal{H})^*$ such that $m(\pi_g^* x \pi_g) = m(x)$ for all $x \in B(\mathcal{H})$, $g \in G$.

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They were able to give partial answers.

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Thank you for your attention!