Weak compactness in Banach lattices

Pedro Tradacete

Universidad Carlos III de Madrid

Based on joint works with A. Avilés, A. J. Guirao, S. Lajara, J. López-Abad, J. Rodríguez

Positivity IX
20 July 2017, Edmonton

1. Weakly compactly generated Banach lattices

2. Shellable weakly compact sets and Talagrand's problem

The promoter

Integration, Vector Measures and Related Topics IV (La Manga del Mar Menor, Spain 2011).

Joe's question: *"Is every Banach lattice that's weakly compactly generated as a Banach lattice a weakly compactly generated Banach space?"*

Some terminology

Definition

Given X Banach lattice, $A \subset X$.

(i) $L(A)$ denotes the smallest (closed) sublattice of X containing A.

(ii) $I(A)$ denotes the smallest (closed) ideal of X containing A.

(iii) $B(A)$ denotes the smallest (closed) band of X containing A.

- Let us denote $A^{\wedge} := \left\{ \bigwedge_{i=1}^{n} a_i : n \in \mathbb{N}, (a_i)_{i=1}^{n} \subset A \right\}$ and $A^{\vee} := \left\{ \bigvee_{i=1}^{n} a_i : n \in \mathbb{N}, (a_i)_{i=1}^{n} \subset A \right\}$. We have
$$L(A) = \overline{span(A)^{\vee\wedge}}$$
- Consider the solid hull $sol(A) = \bigcup_{x \in A} [-|x|, |x|]$. If follows that
$$I(A) = \overline{span}(sol(A)).$$
- If $A^{\perp} = \{x \in X : |x| \wedge |y| = 0 \text{ for every } y \in A\}$, then
$$B(A) = A^{\perp\perp}.$$

Different versions of WCG

Definition

Given X Banach lattice.

(i) X is weakly compactly generated (WCG) if:

$$\exists K \subset X \text{ w.c. such that } X = \overline{span}(K).$$

(ii) X is weakly compactly generated as a lattice (LWCG) if:

$$\exists K \subset X \text{ w.c. such that } X = L(K).$$

(iii) X is weakly compactly generated as an ideal (IWCG) if:

$$\exists K \subset X \text{ w.c. such that } X = I(K).$$

(iv) X is weakly compactly generated as a band (BWCG) if:

$$\exists K \subset X \text{ w.c. such that } X = B(K).$$

$$WCG \Rightarrow LWCG \Rightarrow IWCG \Rightarrow BWCG.$$

Easy facts

Proposition

Banach lattice X with weakly seq. continuous lattice operations.

$$X \text{ LWCG} \Leftrightarrow X \text{ WCG}.$$

Corollary

Let K be a compact Hausdorff topological space. Then:

(i) $C(K)$ is lWCG.

(ii) $C(K)$ LWCG $\Leftrightarrow$ $C(K)$ WCG.

Proposition

Let X be a Banach lattice with the property that the solid hull of any weakly relatively compact set is weakly relatively compact.

$$X \text{ BWCG} \Leftrightarrow X \text{ WCG}.$$

Related counterexamples

Example

ℓ_∞ is IWCG but not WCG (same holds for $C(K)$ with K not Eberlein compact).

Example

For $1 < p < \infty$ the Lorentz space $L_{p,\infty}[0,1]$ is BWCG but not IWCG.

Remark

Suppose X is separable.

1. *X^* is IWCG $\Leftrightarrow$ X^* has a quasi-interior point.*
2. *X^* is BWCG $\Leftrightarrow$ X^* has a weak order unit.*

,

Theorem

Let X be an LWCG Banach lattice. Then $dens(X) = dens(X^*, w^*)$.

Theorem

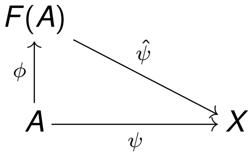
Let X be an order continuous Banach lattice.

$$X \text{ BWCG} \Leftrightarrow X \text{ WCG}.$$

Free Banach lattices

Given a set A, the free Banach lattice generated by A is the (unique) Banach lattice $F(A)$ satisfying

1. there is $\phi : A \to F(A)$ with $\sup_{a\in A} \|\phi(a)\| < \infty$.
2. For every Banach lattice X and $\psi : A \to X$, there is a unique lattice homomorphism $\hat{\psi} : F(A) \to X$ such that $\|\hat{\psi}\| = \sup_{a\in A} \|\psi(a)\|$ and



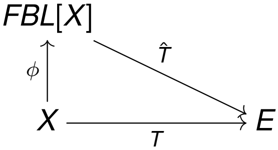
Theorem (De Pagter-Wickstead)

$F(A)$ exists for every A.

The free Banach lattice generated by a Banach space

Let X be a Banach space. Let $FBL[X]$ be the (unique) Banach lattice such that

1. there is a linear isometry $\phi : X \to FBL[X]$,
2. for every Banach lattice E and operator $T : X \to E$ there is a unique lattice homomorphism $\hat{T} : FBL[X] \to E$ such that $\|\hat{T}\| = \|T\|$ and



Theorem (Avilés-Rodríguez-T)

$FBL[X]$ exists for every Banach space X.
Moreover, $F(A) = FBL[\ell_1(A)]$.

Theorem

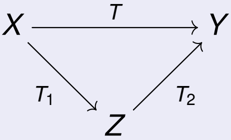
$FBL[\ell_2(\Gamma)]$ is LWCG, but not WCG when Γ is uncountable.

2. Shellable weakly compact sets and Talagrand's problem

Motivation

Theorem (Davis-Figiel-Johnson-Pelczynski 1974)

Given Banach spaces X, Y and a weakly compact operator $T : X \to Y$, there is a reflexive Banach space Z and operators T_1, T_2 such that



Question: If X, Y are Banach lattices, can we make Z a (reflexive) Banach lattice?

Answers:

- Yes, under some conditions (Aliprantis-Burkinshaw 1984).
- In general, NO (Talagrand 1986).

Shellable sets

Theorem (Davis-Figiel-Johnson-Pelczynski)

Let X be a Banach space, $K \subset X$ weakly compact. There is a reflexive Banach space Z and an operator $T : Z \to X$ such that $K \subseteq T(B_Z)$.

Definition

Let X be a Banach space. A weakly compact set $K \subset X$ is *shellable by a reflexive Banach lattice* if there is a **reflexive Banach lattice** E and an operator $T : E \to X$ such that $K \subset T(B_E)$.

Theorem (Aliprantis-Burkinhaw)

Under any of the following assumptions

- *X is a space with an unconditional basis, or*
- *X is a Banach lattice which does not contain c_0,*

every weakly compact set $K \subseteq X$ is shellable by a reflexive Banach lattice.

Talagrand's question

Theorem (Talagrand)

There is a (countable) weakly compact set $K_{\mathcal{T}} \subseteq C[0,1]$ which is not shellable by any reflexive Banach lattice.

$K_{\mathcal{T}}$ is homeomorphic to $\omega^{\omega^2} + 1$.

Question: What is the smallest ordinal α such that there exists a weakly compact set $K \subseteq C[0,1]$ homeomorphic to α which is not shellable by any reflexive Banach lattice?

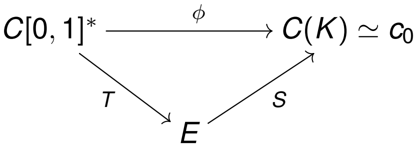
The lower bound

Theorem (López-Abad - T)

Let $K \subseteq C[0,1]$ be a weakly compact set homeomorphic to $\alpha < \omega^\omega$. Then K is shellable by a reflexive Banach lattice.

Sketch of proof:

1. Let $\phi : C[0,1]^* \to C(K)$ be given by $\phi(\mu)(k) = \int k d\mu$.
2. $C(K)$ is isomorphic to c_0.
3. There is a reflexive lattice E such that



4. $\phi^*(\delta_k) = k$ for every $k \in K$.

The upper bound

Consider the Schreier family and its “square”

$$\mathcal{S} = \{s \subset \mathbb{N} : \sharp s \leq \min s\},$$

$$\mathcal{S}_2 = \mathcal{S} \otimes \mathcal{S} = \{\bigcup_{i=1}^{n} s_i : n \leq s_1 < \ldots < s_n,\ s_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}.$$

$S, S_2 \subset \mathcal{P}^{<\infty}(\mathbb{N})$ are compact and homeomorphic to $\omega^\omega + 1$ and $\omega^{\omega^2} + 1$ respectively.
Each element $s \in \mathcal{S}_2$ has a unique decomposition

$$s = s[0] \cup s[1] \cdots \cup s[n],$$

where $s[0] < s[1] < \cdots < s[n]$, $\{\min s[i]\}_{i \leq n} \in \mathcal{S}$, $s[n] \in \mathcal{S}$ and $\min s[i] = \sharp s[i]$ for $0 \leq i < n$.

The upper bound

Given $s = \{m_0 < \cdots < m_k\} \in \mathcal{S}$ and $t = t[0] \cup \cdots \cup t[l] \in \mathcal{S}_2$ let

$$\langle s, t\rangle = \#(\{0 \leq i \leq \min\{k, l\} \ : \ m_i \in t[i]\}).$$

$$\Theta(s, t) = \langle s, t\rangle + 1 \qquad (mod\, 2).$$

Let $\Theta_0 : \mathcal{S} \to C(\mathcal{S}_2)$ be the mapping that for $s = \{m_0 < \cdots < m_k\} \in \mathcal{S}$ for every $t = t[0] \cup \cdots \cup t[l] \in \mathcal{S}_2$,

$$\Theta_0(s)(t) = \Theta(s, t).$$

$\Theta_0 : \mathcal{S} \to C(\mathcal{S}_2)$ is well-defined and (weakly-)continuous.
Let $K_\omega := \Theta_0(\mathcal{S}) \subseteq C(\mathcal{S}_2)$ is weakly compact and homeomorphic to $\omega^\omega + 1$ (and extending its elements by zero we get $K_\omega \subset C[0, 1]$).

Theorem (López-Abad - T)

$K_\omega \subset C(\mathcal{S}_2)$ is not shellable by any reflexive Banach lattice.

- A. Avilés, A. J. Guirao, S. Lajara, J. Rodríguez, P. Tradacete, Weakly compactly generated Banach lattices. Studia Math. 234 (2016), no. 2, 165–183.
- A. Avilés, J. Rodríguez, P. Tradacete, The free Banach lattice generated by a Banach space. https://arxiv.org/pdf/1706.08147
- J. López-Abad, P. Tradacete, Shellable weakly compact subsets of $C[0,1]$. Math. Ann. 367 (2017), no. 3-4, 1777–1790.

Thank you for your attention!