Weak compactness in Banach lattices

Pedro Tradacete

Universidad Carlos III de Madrid

Based on joint works with A. Avilés, A. J. Guirao, S. Lajara, J. López-Abad, J. Rodríguez

Positivity IX
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1. Weakly compactly generated Banach lattices

2. Shellable weakly compact sets and Talagrand’s problem
Integration, Vector Measures and Related Topics IV (La Manga del Mar Menor, Spain 2011).

Joe’s question: “Is every Banach lattice that’s weakly compactly generated as a Banach lattice a weakly compactly generated Banach space?”
Some terminology

Definition

Given $X$ Banach lattice, $A \subset X$.

(i) $L(A)$ denotes the smallest (closed) sublattice of $X$ containing $A$.

(ii) $I(A)$ denotes the smallest (closed) ideal of $X$ containing $A$.

(iii) $B(A)$ denotes the smallest (closed) band of $X$ containing $A$.

Let us denote $A^\wedge := \left\{ \bigwedge_{i=1}^{n} a_i : n \in \mathbb{N}, (a_i)_{i=1}^{n} \subset A \right\}$ and $A^\vee := \left\{ \bigvee_{i=1}^{n} a_i : n \in \mathbb{N}, (a_i)_{i=1}^{n} \subset A \right\}$. We have

$$L(A) = \overline{\text{span}(A)^{\wedge\vee}}$$

Consider the solid hull $\text{sol}(A) = \bigcup_{x \in A} [-|x|, |x|]$. It follows that

$$I(A) = \overline{\text{span}(\text{sol}(A))}.$$

If $A^\perp = \{ x \in X : |x| \wedge |y| = 0 \text{ for every } y \in A \}$, then

$$B(A) = A^{\perp\perp}.$$
Different versions of WCG

Definition

Given $X$ Banach lattice.

(i) $X$ is weakly compactly generated (WCG) if:
\[ \exists K \subset X \text{ w.c. such that } X = \text{span}(K). \]

(ii) $X$ is weakly compactly generated as a lattice (LWCG) if:
\[ \exists K \subset X \text{ w.c. such that } X = L(K). \]

(iii) $X$ is weakly compactly generated as an ideal (IWCG) if:
\[ \exists K \subset X \text{ w.c. such that } X = I(K). \]

(iv) $X$ is weakly compactly generated as a band (BWCG) if:
\[ \exists K \subset X \text{ w.c. such that } X = B(K). \]

\[ \text{WCG} \Rightarrow \text{LWCG} \Rightarrow \text{IWCG} \Rightarrow \text{BWCG}. \]
Easy facts

**Proposition**

*Banach lattice* $X$ *with weakly seq. continuous lattice operations.*

$$X \text{ LWCG} \iff X \text{ WCG}.$$ 

**Corollary**

Let $K$ be a compact Hausdorff topological space. Then:

(i) $C(K)$ is IWCG.

(ii) $C(K) \text{ LWCG} \iff C(K) \text{ WCG}.$

**Proposition**

Let $X$ be a Banach lattice with the property that the solid hull of any weakly relatively compact set is weakly relatively compact.

$$X \text{ BWCG} \iff X \text{ WCG}.$$
Related counterexamples

Example

\( \ell_\infty \) is IWCG but not WCG (same holds for \( C(K) \) with \( K \) not Eberlein compact).

Example

For \( 1 < p < \infty \) the Lorentz space \( L_{p,\infty}[0,1] \) is BWCG but not IWCG.

Remark

Suppose \( X \) is separable.

1. \( X^* \) is IWCG \( \iff \) \( X^* \) has a quasi-interior point.
2. \( X^* \) is BWCG \( \iff \) \( X^* \) has a weak order unit.
Theorem

Let $X$ be an LWCG Banach lattice. Then $\text{dens}(X) = \text{dens}(X^*, w^*)$.

Theorem

Let $X$ be an order continuous Banach lattice.

$X \text{ BWCG} \iff X \text{ WCG}$. 
Free Banach lattices

Given a set $A$, the free Banach lattice generated by $A$ is the (unique) Banach lattice $F(A)$ satisfying

1) there is $\phi : A \to F(A)$ with $\sup_{a \in A} \| \phi(a) \| < \infty$.

2) For every Banach lattice $X$ and $\psi : A \to X$, there is a unique lattice homomorphism $\hat{\psi} : F(A) \to X$ such that $\| \hat{\psi} \| = \sup_{a \in A} \| \psi(a) \|$ and

\[
\begin{array}{ccc}
F(A) & \xrightarrow{\hat{\psi}} & X \\
\phi \downarrow & & \downarrow \psi \\
A & \xrightarrow{\psi} & X
\end{array}
\]

Theorem (De Pagter-Wickstead)

$F(A)$ exists for every $A$. 
The free Banach lattice generated by a Banach space

Let $X$ be a Banach space. Let $FBL[X]$ be the (unique) Banach lattice such that

1. there is a linear isometry $\phi : X \to FBL[X],$

2. for every Banach lattice $E$ and operator $T : X \to E$ there is a unique lattice homomorphism $\hat{T} : FBL[X] \to E$ such that $\|\hat{T}\| = \|T\|$ and

$$FBL[X] \xrightarrow{\phi} X \xrightarrow{T} E \xrightarrow{\hat{T}} E$$

Theorem (Avilés-Rodríguez-T)

$FBL[X]$ exists for every Banach space $X$.
Moreover, $F(A) = FBL[\ell_1(A)].$

Theorem

$FBL[\ell_2(\Gamma)]$ is LWCG, but not WCG when $\Gamma$ is uncountable.
2. Shellable weakly compact sets and Talagrand’s problem
Motivation

Theorem (Davis-Figiel-Johnson-Pelczynski 1974)

Given Banach spaces $X$, $Y$ and a weakly compact operator $T : X \to Y$, there is a reflexive Banach space $Z$ and operators $T_1$, $T_2$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{T_1} & & \downarrow{T_2} \\
Z & & 
\end{array}
$$

Question: If $X$, $Y$ are Banach lattices, can we make $Z$ a (reflexive) Banach lattice?

Answers:

- Yes, under some conditions (Aliprantis-Burkinshaw 1984).
- In general, NO (Talagrand 1986).
Shellable sets

**Theorem (Davis-Figiel-Johnson-Pelczynski)**

Let $X$ be a Banach space, $K \subset X$ weakly compact. There is a reflexive Banach space $Z$ and an operator $T : Z \to X$ such that $K \subseteq T(B_Z)$.

**Definition**

Let $X$ be a Banach space. A weakly compact set $K \subset X$ is shellable by a reflexive Banach lattice if there is a reflexive Banach lattice $E$ and an operator $T : E \to X$ such that $K \subseteq T(B_E)$.

**Theorem (Aliprantis-Burkinshaw)**

Under any of the following assumptions

- $X$ is a space with an unconditional basis, or
- $X$ is a Banach lattice which does not contain $c_0$,

every weakly compact set $K \subset X$ is shellable by a reflexive Banach lattice.
Talagrand’s question

**Theorem (Talagrand)**

There is a (countable) weakly compact set $K_T \subseteq C[0, 1]$ which is not shellable by any reflexive Banach lattice.

$K_T$ is homeomorphic to $\omega^{\omega^2} + 1$.

**Question:** What is the smallest ordinal $\alpha$ such that there exists a weakly compact set $K \subseteq C[0, 1]$ homeomorphic to $\alpha$ which is not shellable by any reflexive Banach lattice?
The lower bound

**Theorem (López-Abad - T)**

Let \( K \subseteq C[0, 1] \) be a weakly compact set homeomorphic to \( \alpha < \omega^\omega \). Then \( K \) is shellable by a reflexive Banach lattice.

**Sketch of proof:**

1. Let \( \phi : C[0, 1]^* \to C(K) \) be given by \( \phi(\mu)(k) = \int k \, d\mu \).
2. \( C(K) \) is isomorphic to \( c_0 \).
3. There is a reflexive lattice \( E \) such that

\[
C[0, 1]^* \xrightarrow{\phi} C(K) \cong c_0
\]

\[
C[0, 1]^* \xrightarrow{T} E \xrightarrow{S} C(K) \cong c_0
\]

4. \( \phi^*(\delta_k) = k \) for every \( k \in K \).
The upper bound

Consider the Schreier family and its “square”

\[ S = \{ s \subset \mathbb{N} : \#s \leq \min s \}, \]

\[ S_2 = S \otimes S = \bigcup_{i=1}^{n} s_i : n \leq s_1 < \ldots < s_n, s_i \in S \text{ for } 1 \leq i \leq n \}. \]

\( S, S_2 \subset \mathcal{P}_{<\infty}(\mathbb{N}) \) are compact and homeomorphic to \( \omega^\omega + 1 \) and \( \omega^{\omega^2} + 1 \) respectively.

Each element \( s \in S_2 \) has a unique decomposition

\[ s = s[0] \cup s[1] \cdots \cup s[n], \]

where \( s[0] < s[1] < \cdots < s[n] \), \( \{ \min s[i] \}_{i \leq n} \in S \), \( s[n] \in S \) and \( \min s[i] = \#s[i] \) for \( 0 \leq i < n \).
The upper bound
Given \( s = \{ m_0 < \cdots < m_k \} \in S \) and \( t = t[0] \cup \cdots \cup t[l] \in S_2 \) let

\[
\langle s, t \rangle = \#(\{ 0 \leq i \leq \min\{ k, l \} : m_i \in t[i] \} ).
\]

\[
\Theta(s, t) = \langle s, t \rangle + 1 \quad (\text{mod} \ 2).
\]

Let \( \Theta_0 : S \to C(S_2) \) be the mapping that for \( s = \{ m_0 < \cdots < m_k \} \in S \) for every \( t = t[0] \cup \cdots \cup t[l] \in S_2 \),

\[
\Theta_0(s)(t) = \Theta(s, t).
\]

\( \Theta_0 : S \to C(S_2) \) is well-defined and (weakly-)continuous.
Let \( K_\omega := \Theta_0(S) \subseteq C(S_2) \) is weakly compact and homeomorphic to \( \omega^\omega + 1 \) (and extending its elements by zero we get \( K_\omega \subset C[0, 1] \)).

**Theorem (López-Abad - T)**

\( K_\omega \subset C(S_2) \) is not shellable by any reflexive Banach lattice.


Thank you for your attention!