The almost-invariant subspace problem for Banach spaces

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- Read - bounded operator on ℓ_1 without invariant subspaces
- Argyros and Haydon - example of a Banach space such that every bounded operator is a compact perturbation of a multiple of identity
- question still open for l_2, reflexive Banach spaces, dual operators, positive operators, etc...

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For Hilbert spaces: Does there exist Y infinite dimensional and with infinite dimensional orthogonal complement such that $(I - P)TP$ is finite rank (P is the orthogonal projection onto Y)?

Equivalently, does there exist Y infinite dimensional and with infinite dimensional orthogonal complement $Y^{\perp}$ such that for the decomposition $\mathcal{H} = Y \oplus Y^{\perp}$ we have $T = \begin{bmatrix} * & * \\ F & * \end{bmatrix}$ with F finite rank?

Related results

Theorem(Brown, Pearcy, 1971)

Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists a scalar λ and a decomposition of $\mathcal{H} = Y \oplus Y^{\perp}$ into infinite dimensional subspaces such that the corresponding matrix representation of T has the form $T = \begin{bmatrix} \lambda I + K & * \\ F & * \end{bmatrix}$ where K and F are compact and have norms at most ε.

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In particular, for any $T \in \mathcal{B}(\mathcal{H})$ there exists Y infinite dimensional with infinite dimensional orthogonal complement such that Y is invariant under $T - F$, where $F := (I - P)TP$ is compact.

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Theorem(Voiculescu, 1976)

Under the same hypotheses, T has the form $T = \begin{bmatrix} * & F_2 \\ F_1 & * \end{bmatrix}$ where F_1 and F_2 are compact with norms at most ε.

An equivalent formulation: almost-invariant half-spaces

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If X is a Banach space, $T \in \mathcal{B}(X)$ and Y is a subspace of X, then Y is called **almost invariant** for T, or T-**almost invariant** if there exists a finite dimensional subspace M of X such that $T(Y) \subseteq Y + M$.

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Almost invariant half-space problem

Does every bounded linear operator on a Banach space have almost invariant half-spaces?

Almost invariant half-space problem

Proposition(APTT, 2009)

Let $T \in \mathcal{B}(X)$ and $Y \subseteq X$ be a half-space. Then Y is almost invariant under T if and only if Y is invariant under $T + F$ for some finite rank operator F.

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Let T be an operator on a Banach space X. If T has an almost invariant half-space then so does its adjoint T^*.

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Let X be reflexive and $T \in \mathcal{B}(X)$ be such that one of T or T^* has a boundary point of the spectrum which is not an eigenvalue. Then T admits an almost-invariant half-space with error at most one.

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Note that an operator $T \in \mathcal{B}(X)$ which has no invariant subspaces cannot have any eigenvalues. It follows from the previous theorem that such an operator has an almost-invariant half-space.In particular, all known counterexamples to the invariant subspace problem (e.g. the operators constructed by Enflo or Read) are not counterexamples to the almost-invariant half-space problem.

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When $\partial\sigma(T) \setminus \sigma_p(T) = \emptyset$ and $\partial\sigma(T^*) \setminus \sigma_p(T^*) = \emptyset$: an important ingredient is the main theorem from APTT(2009).

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For Hilbert spaces:

Corollary

For any $T \in \mathcal{B}(\mathcal{H})$ there exist an infinite dimensional subspace Y with infinite dimensional orthogonal complement such that $(I - P)TP$ has rank at most one, where P is the orthogonal projection onto Y.

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Equivalently, relative to the decomposition $\mathcal{H} = Y \oplus Y^{\perp}$, T has the form $T = \begin{bmatrix} * & * \\ F & * \end{bmatrix}$ where F has rank one.

Results: Perturbations of small norm

Theorem (PT, 2013)

Let $T \in \mathcal{B}(X)$ such that there exists $\lambda \in \partial\sigma(T)$ which is not an eigenvalue. Then for any $\varepsilon > 0$, T has an almost invariant half-space Y_ε such that $(T - \lambda I)_{|Y_\varepsilon}$ is compact and $\|(T - \lambda I)_{|Y_\varepsilon}\| < \varepsilon$

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Theorem(Brown, Pearcy, 1971)

Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists a scalar λ and a decomposition of $\mathcal{H} = Y \oplus Y^{\perp}$ into infinite dimensional subspaces such that the corresponding matrix representation of T has the form $T = \begin{bmatrix} \lambda I + K & * \\ F & * \end{bmatrix}$ where K and F are compact and have norms at most ε.

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Theorem (T, Wallis, 2017)

Let X be a reflexive Banach space. Then there exists $d \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is an operator $F \in \mathcal{B}(X)$ of rank $\leq d$ satisfying $\|F\| < \varepsilon$, and such that $T + F$ admits an IHS.

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Let X be a Banach space and $T \in \mathcal{B}(X)$ a bounded operator such that $\partial\sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$. Then for any $\varepsilon > 0$ there exists a rank one operator F with $\|F\| < \varepsilon$ such that $T + F$ has an invariant half-space.

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Let X be a Banach space and $T \in \mathcal{B}(X)$ a bounded operator. Then for any $\varepsilon > 0$ there exists a finite rank operator F with $\|F\| < \varepsilon$ such that $T + F$ has an invariant half-space. Moreover, if $\partial\sigma(T) \setminus \sigma_p(T) \neq \emptyset$ or $\partial\sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$, F can be taken to be rank one.

Some open problems

Theorem(Voiculescu, 1976)

$T \in B(H)$ has the form $T = \begin{bmatrix} * & F_2 \\ F_1 & * \end{bmatrix}$ where F_1 and F_2 are compact with norms at most ε.

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In other words, there exist K compact such that $T - K$ has a reducing half-space.

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In other words, there exist K compact such that $T - K$ has a reducing half-space.
Question: Can we take K finite rank?

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Question: Can we also get F rank one when $\partial\sigma(T) \setminus \sigma_p(T) = \emptyset$?

The Method (sketch)

For a nonzero vector $e \in X$ and for $\lambda \in \mathbb{C} \backslash \sigma(T)$ define a vector h_λ in X by

$$h_\lambda := \left(\lambda I - T\right)^{-1}(e).$$

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Hence, for a subset $A \subset \mathbb{C} \backslash \sigma(T)$, the closed subspace Y of X defined by

$$Y = \overline{\operatorname{span}}\big\{h_\lambda \colon \lambda \in A\big\}$$

is a T-almost invariant subspace (which is not not necessarily a half-space).

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1. S fails to contain a basic sequence.
2. $\overline{S}^{weak}$ is weakly compact and $0 \notin \overline{S}^{weak}$.

The Method (sketch)

For the non-reflexive case an important ingredient is the following theorem.

Theorem (Johnson, Rosenthal, 1972)

If (x_n^*) is a semi-normalized, w^*-null, sequence in a dual Banach space X^*, then there exists a a basic subsequence (y_n^*) of (x_n^*), and a bounded sequence (y_n) in X such that $y_i^*(y_j) = \delta_{ij}$ for all $1 \leq i, j < \infty$.

Thank you!