The almost-invariant subspace problem for Banach spaces

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Motivation

Invariant subspace problem

Does every bounded linear operator acting on a separable (complex) Banach space have a closed non-trivial invariant subspace?

Aronszajn and Smith - for compact operators
Lomonosov - for operators commuting with a compact operator
Enflo - first example of a bounded operator without invariant subspaces
Read - bounded operator on $\ell_1$
Argyros and Haydon - example of a Banach space such that every bounded operator is a compact perturbation of a multiple of identity

question still open for $l_2$, reflexive Banach spaces, dual operators, positive operators, etc...
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A closely related question

**Question**

Given a bounded linear operator $T$ acting on a complex Banach space $X$, can we perturb it by a finite rank operator $F$ such that $T + F$ has an invariant subspace of infinite dimension and codimension in $X$?

For Hilbert spaces: Does there exist $Y$, infinite dimensional and with infinite dimensional orthogonal complement such that $(I - P)TP$ is finite rank ($P$ is the orthogonal projection onto $Y$)?

Equivalently, does there exist $Y$, infinite dimensional and with infinite dimensional orthogonal complement $Y^\perp$ such that for the decomposition $H = Y \oplus Y^\perp$ we have $T = \begin{bmatrix} * & * \\ * & F \end{bmatrix}$ with $F$ finite rank?
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Theorem (Brown, Pearcy, 1971)

Let $T \in B(\mathcal{H})$ and $\varepsilon > 0$. Then there exists a scalar $\lambda$ and a decomposition of $\mathcal{H} = Y \oplus Y^\perp$ into infinite dimensional subspaces such that the corresponding matrix representation of $T$ has the form $T = \begin{bmatrix} \lambda I + K & * \\ F & * \end{bmatrix}$ where $K$ and $F$ are compact and have norms at most $\varepsilon$. In particular, for any $T \in B(\mathcal{H})$ there exists $Y$ infinite dimensional with infinite dimensional orthogonal complement such that $Y$ is invariant under $T - F$, where $F := (I - P)TP$ is compact.
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Related results

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**Theorem (Voiculescu, 1976)**

Under the same hypotheses, $T$ has the form $T = \begin{bmatrix} * & F_2 \\ F_1 & * \end{bmatrix}$ where $F_1$ and $F_2$ are compact with norms at most $\varepsilon$. 
An equivalent formulation: almost-invariant half-spaces

An equivalent formulation of this problem was first introduced in a paper by Androulakis, Popov, T., Troitsky in 2009.
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Definitions (Androulakis, Popov, T., Troitsky, 2009)

If $X$ is a Banach space, $T \in B(X)$ and $Y$ is a subspace of $X$, then $Y$ is called **almost invariant** for $T$, or **$T$-almost invariant** if there exists a finite dimensional subspace $M$ of $X$ such that $T(Y) \subseteq Y + M$. A subspace $Y$ of a Banach space $X$ is called **half-space** if it is of both infinite dimension and infinite codimension in $X$. **Almost invariant half-space problem**

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**Proposition (APTT, 2009)**

Let $T$ be an operator on a Banach space $X$. If $T$ has an almost invariant half-space then so does its adjoint $T^*$. 
Theorem (APTT, 2009)

Let $X$ be a Banach space and $T \in \mathcal{B}(X)$ satisfy the following:

1. $T$ has no eigenvalues.
2. The unbounded component of the resolvent contains $\{0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.
3. There is a vector whose orbit is a minimal sequence.

Then $T$ has an almost invariant half-space with at most 1-dimensional "error".

Theorem was used to show existence of invariant half-spaces for weighted shifts on $l^p$.

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Let $T \in \mathcal{B}(X)$ be such that $\partial \sigma(T) \setminus \sigma_p(T) \neq \emptyset$. Then $T$ admits an almost-invariant half-space with error at most one.

### Corollary
Let $X$ be reflexive and $T \in \mathcal{B}(X)$ be such that one of $T$ or $T^*$ has a boundary point of the spectrum which is not an eigenvalue. Then $T$ admits an almost-invariant half-space with error at most one.
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Note that an operator $T \in B(X)$ which has no invariant subspaces cannot have any eigenvalues.
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Note that an operator $T \in \mathcal{B}(X)$ which has no invariant subspaces cannot have any eigenvalues. It follows from the previous theorem that such an operator has an almost-invariant half-space. In particular, all known counterexamples to the invariant subspace problem (e.g. the operators constructed by Enflo or Read) are not counterexamples to the almost-invariant half-space problem.
Let $X$ be reflexive and $T \in \mathcal{B}(X)$. Then $T$ admits an almost-invariant half-space with error at most one.
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When $\partial \sigma(T) \setminus \sigma_p(T) = \emptyset$ and $\partial \sigma(T^*) \setminus \sigma_p(T^*) = \emptyset$: an important ingredient is the main theorem from APTT(2009).
For Hilbert spaces:

**Corollary**

For any $T \in \mathcal{B}(\mathcal{H})$ there exist an infinite dimensional subspace $Y$ with infinite dimensional orthogonal complement such that $(I - P)TP$ has rank at most one, where $P$ is the orthogonal projection onto $Y$. 
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Equivalently, relative to the decomposition $\mathcal{H} = Y \oplus Y^\perp$, $T$ has the form $T = \begin{bmatrix} * & * \\ F & * \end{bmatrix}$ where $F$ has rank one.
Results: Perturbations of small norm

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Let $T \in B(X)$ such that there exists $\lambda \in \partial \sigma(T)$ which is not an eigenvalue. Then for any $\varepsilon > 0$, $T$ has an almost invariant half-space $Y_\varepsilon$ such that $(T - \lambda I)|_{Y_\varepsilon}$ is compact and $\|(T - \lambda I)|_{Y_\varepsilon}\| < \varepsilon$
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$T = \begin{bmatrix} \lambda I + K & * \\ F & * \end{bmatrix}$

where $K$ is compact, $F$ has rank one, and both have norms at most $\varepsilon$. 
Theorem (Brown, Pearcy, 1971)

Let $T \in B(H)$ and $\varepsilon > 0$. Then there exists a scalar $\lambda$ and a decomposition of $H = Y \oplus Y^\perp$ into infinite dimensional subspaces such that the corresponding matrix representation of $T$ has the form

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Let $X$ be a reflexive Banach space. Then there exists $d \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is an operator $F \in \mathcal{B}(X)$ of rank $\leq d$ satisfying $\|F\| < \varepsilon$, and such that $T + F$ admits an IHS.
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**Theorem (T, preprint, 2017)**

Let $X$ be a Banach space and $T \in \mathcal{B}(X)$ a bounded operator such that $\partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$. Then for any $\varepsilon > 0$ there exists a rank one operator $F$ with $\|F\| < \varepsilon$ such that $T + F$ has an invariant half-space.
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Let $X$ be a Banach space and $T \in B(X)$ a bounded operator. Then for any $\varepsilon > 0$ there exists a finite rank operator $F$ with $\|F\| < \varepsilon$ such that $T + F$ has an invariant half-space. Moreover, if $\partial \sigma(T) \setminus \sigma_p(T) \neq \emptyset$ or $\partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$, $F$ can be taken to be rank one.
Some open problems

**Theorem (Voiculescu, 1976)**

\[ T \in B(H) \text{ has the form } T = \begin{bmatrix} * & F_2 \\ F_1 & * \end{bmatrix} \]

where \( F_1 \) and \( F_2 \) are compact with norms at most \( \varepsilon \).

In other words, there exist \( K \) compact such that \( T - K \) has a reducing half-space.

Question: Can we take \( K \) finite rank?
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**Theorem (Popov, T, 2013)**

If $\lambda \in \partial \sigma(T) \setminus \sigma_p(T)$, then for any $\varepsilon > 0$, $T$ has the form $T = \begin{bmatrix} \lambda I + K & * \\ F & * \end{bmatrix}$ where $K$ is compact, $F$ has rank one, and both have norms at most $\varepsilon$. 

Question: Can we also get $F$ rank one when $\partial \sigma(T) \setminus \sigma_p(T) = \emptyset$?
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Question: Can we also get \( F \) rank one when \( \partial \sigma(T) \setminus \sigma_p(T) = \emptyset \)?
For a nonzero vector $e \in X$ and for $\lambda \in \mathbb{C}\backslash \sigma(T)$ define a vector $h_\lambda$ in $X$ by

$$h_\lambda := (\lambda I - T)^{-1}(e).$$
The Method (sketch)

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Hence, for a subset $A \subset \mathbb{C} \setminus \sigma(T)$, the closed subspace $Y$ of $X$ defined by

$$Y = \overline{\text{span}\{h_\lambda : \lambda \in A\}}$$

is a $T$-almost invariant subspace (which is not necessarily a half-space).
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If \((x_n)_n\) is a \textit{basic sequence} then \(\overline{\text{span}}\{x_{2n}\}_n\) is a half subspace of \(\overline{\text{span}}\{x_n\}_n\).
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We try to find \(e \in X\) and a sequence \((\lambda_n)_n\) in the resolvent such that \((h_{\lambda_n})_n\) is basic sequence.
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Old criterion for extracting basic sequences:
If \((x_n)_n\) is a basic sequence then \(\text{span}\{x_{2n}\}_n\) is a half subspace of \(\text{span}\{x_n\}_n\).

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**Theorem (Kadets, Pełczyński, 1965)**

Let \(S\) be a bounded subset of a Banach space \(X\) such that \(0 \not\in \overline{S}_{\|\cdot\|}\). Then the following are equivalent:
The Method (sketch)

If \((x_n)_n\) is a \textit{basic sequence} then \(\overline{\text{span}}\{x_{2n}\}_n\) is a half subspace of \(\overline{\text{span}}\{x_n\}_n\).

We try to find \(e \in X\) and a sequence \((\lambda_n)_n\) in the resolvent such that \((h_{\lambda_n})_n\) is basic sequence.

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**Theorem (Kadets, Pełczyński, 1965)**

Let \(S\) be a bounded subset of a Banach space \(X\) such that \(0 \notin \overline{S^{||\cdot||}}\). Then the following are equivalent:

1. \(S\) fails to contain a basic sequence.
If \((x_n)_n\) is a basic sequence then \(\overline{\text{span}}\{x_{2n}\}_n\) is a half subspace of \(\overline{\text{span}}\{x_n\}_n\).

We try to find \(e \in X\) and a sequence \((\lambda_n)_n\) in the resolvent such that \((h_{\lambda_n})_n\) is basic sequence.

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**Theorem (Kadets, Pełczyński, 1965)**

Let \(S\) be a bounded subset of a Banach space \(X\) such that \(0 \notin \overline{S}\). Then the following are equivalent:

1. \(S\) fails to contain a basic sequence.
2. \(\overline{S}^{\text{weak}}\) is weakly compact and \(0 \notin \overline{S}^{\text{weak}}\).
For the non-reflexive case an important ingredient is the following theorem.

**Theorem (Johnson, Rosenthal, 1972)**

If \((x_n^*)\) is a semi-normalized, \(w^*\)-null, sequence in a dual Banach space \(X^*\), then there exists a a basic subsequence \((y_n^*)\) of \((x_n^*)\), and a bounded sequence \((y_n)\) in \(X\) such that \(y_i^*(y_j) = \delta_{ij}\) for all \(1 \leq i, j < \infty\).
Thank you!