Unbounded topologies and uo-convergence

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July 20, 2017

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If τ is a locally solid topology, it has a base, $\{V_i\}$, at zero consisting of solid sets. The collection $\{(V_i)_u\}$ where $u \in X_+$ defines a locally solid topology, $u\tau$, on X.

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- $x_\alpha \xrightarrow{u\tau} 0$ iff $|x_\alpha| \wedge u \xrightarrow{\tau} 0$ for all $u \in X_+$
- The map $\tau \mapsto u\tau$ from the set of locally solid topologies on X to itself is idempotent

Definition

A locally solid topology is **unbounded** if $\tau = u\tau$ or, equivalently, if $\tau = u\sigma$ for some locally solid topology σ.

What is more important u or τ?

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Lemma

Let X be a vector lattice, $u \in X_+$ and V a solid subset of X. Then V_u is either contained in $[-u, u]$ or contains a non-trivial ideal. If V is, further, absorbing, and V_u is contained in $[-u, u]$, then u is a strong unit.

Theorem (Kandic, Marabeh, Troitsky)

Let X be an order continuous Banach lattice. The un-topology is locally convex iff X is atomic. In general, if $0 \neq \varphi \in (X, un)^*$ then φ is a linear combination of the coordinate functionals of finitely many atoms.

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Theorem

Let (X, τ) be an order continuous locally solid vector lattice. The $u\tau$-topology is locally convex iff X is atomic. In general, if $0 \neq \varphi \in (X, u\tau)^*$ then φ is a linear combination of the coordinate functionals of finitely many atoms.

Notice, no metrizability, local convexity, or completeness of the topology τ is needed in the previous theorem. This is a general phenomenon.

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Many results of the *un*-papers carry over to the locally solid setting by simply replacing "Banach lattice" by "(Hausdorff) locally solid topology".

A partial reason for this is that associated to an unbounded topology, σ, are many topologies τ satisfying $u\tau = \sigma$ - not all of these topologies are as "nice" as that of a complete lattice norm.

An application of unbounded topologies

Recall,

Theorem (Amemiya-Mori)

All Hausdorff order continuous topologies on a vector lattice X induce the same topology on the order bounded subsets of X.

A use for unbounded topologies

Lemma (Gao, Troitsky, Xanthos)

Let X be a vector lattice, and Y a sublattice of X. Then Y is uo-closed in X if and only if it is o-closed in X.

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Let X be a vector lattice, τ and σ Hausdorff order continuous topologies on X, and Y a sublattice of X. Y is τ-closed in X if and only if it is σ-closed in X.

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Proof.

Suppose Y is τ-closed; then Y is $u\tau$-closed. Suppose $(y_\alpha) \subseteq Y$ and $y_\alpha \xrightarrow{u\sigma} x$. This means that $|y_\alpha - x| \wedge u \xrightarrow{\sigma} 0$ for all $u \in X_+$. Since $(|y_\alpha - x| \wedge u)$ is order bounded, this is equivalent to $|y_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for all $u \in X_+$, which means $y_\alpha \xrightarrow{u\tau} x$. Therefore, $x \in Y$ and Y is $u\sigma$-closed. This implies Y is σ-closed. $\square$

The picture is clearer with general topologies

It was proved in [KMT] that if X is an order continuous Banach lattice then the *un*-topology is complete iff X is finite-dimensional. Can we explain why this is true?

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Corollary

Let τ be a Hausdorff order continuous topology on a vector lattice X. τ is complete iff X is universally complete.

Liftings to the universal completion

Theorem

For a Hausdorff order continuous topology τ on X, TFAE:

1. τ *extends to a Hausdorff order continuous topology on* X^u*;*
2. τ *extends to a locally solid topology on* X^u*;*
3. *The topological completion* $\widehat{X}$ *of* (X, τ) *is lattice isomorphic to* X^u*, that is,* $\widehat{X}$ *is the universal completion of* X*;*

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4. τ *is unbounded.*

Minimal topologies

Definition

A Hausdorff locally solid topology on X is **minimal** if there is no coarser Hausdorff locally solid topology on X. It is **least** if it is coarser than every locally solid topology on X.

Theorem (Labuda, Conradie)

Minimal topologies are order continuous and unique.

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Theorem (Aliprantis and Burkinshaw)

If (X, Σ, μ) is a σ-finite measure space, then for each $0 \leq p < \infty$ the topology of (local) convergence in measure on $L_p(\mu)$ is the least topology. L_∞ does not admit a least topology; convergence in measure is the minimal topology on L_∞.

Connection between *uo*, $u\tau$ and minimal topologies

Theorem

Let τ be a Hausdorff locally solid topology on X. TFAE:

1. *uo-null nets are τ-null*
2. *τ is order continuous and unbounded*
3. *τ is minimal*

The equivalence of (i) and (iii) generalizes a classical relation between convergence a.e. and convergence in measure to vector lattices!