ORDER ISOMORPHISMS OF OPERATOR INTERVALS

Peter Šemrl
University of Ljubljana

H Hilbert space, $S(H)$ the set of all linear bounded self-adjoint operators on H

The usual partial order on $S(H)$:

$$A \leq B \iff$$
$$\langle Ax, x \rangle \leq \langle Bx, x \rangle \text{ for every } x \in H$$

Mathematical foundations of quantum mechanics: linear bounded self-adjoint operators $\equiv$ bounded observables, $A \leq B \iff$ the mean value (expectation) of A in every state is less than or equal to the mean value of B in the same state

THEOREM (Molnár 2001). $\phi : S(H) \to S(H)$ bijective map such that

$$A \le B \iff \phi(A) \le \phi(B).$$

Then ϕ is of the form

$$\phi(A) = TAT^* + B, \quad A \in S(H).$$

Here, $B \in S(H)$, $T : H \to H$ bdd linear or conjugate-linear bijective operator.

We will restrict to the finite-dimensional case.

H_n the set of all $n \times n$ hermitian matrices

$$A = A^*$$

$A = UDU^*$, D diagonal matrix with real entries on the main diagonal (eigenvalues of A)

$$A \geq 0 \iff$$

all eigenvalues of A are non-negative.

$$A \leq B \iff B - A \geq 0$$

Molnár's theorem again, this time just the finite-dimensional case:

THEOREM $\phi : H_n \to H_n$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exist an invertible matrix T and $B \in H_n$ such that either

$$\phi(A) = TAT^* + B$$

for every $A \in H_n$, or

$$\phi(A) = TA^{tr}T^* + B$$

for every $A \in H_n$.

Effect algebra E_n:

$$E_n = \{A \in H_n \,:\, 0 \leq A \leq I\}$$

Orthocomplementation on E_n:

$$A \in E_n \;:\quad A^{\perp} = I - A$$

THEOREM (Ludwig, characterization of ortho-order automorphisms of E_n).
$\phi : E_n \to E_n$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B)$$

and

$$\phi(A^{\perp}) = \phi(A)^{\perp}.$$

Then there exists a unitary matrix U such that either

$$\phi(A) = UAU^*$$

for every $A \in E_n$, or

$$\phi(A) = UA^{tr}U^*$$

for every $A \in E_n$.

Molnár: bijectivity + order preserving

Ludwig: bijectivity + order preserving + orthocomplementation preserving

CONJECTURE. $\phi : E_n \to E_n$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exists a unitary matrix U such that either

$$\phi(A) = UAU^*$$

for every $A \in E_n$, or

$$\phi(A) = UA^{tr}U^*$$

for every $A \in E_n$.

Wrong!

Example:

$$A \mapsto$$

$$S^{-1/2}\left((I - T^2 + T(I+A)^{-1}T)^{-1} - I\right) S^{-1/2}$$

$$S = \frac{T^2}{2I - T^2}$$

Operator intervals: $A, B \in H_n$, $A < B$
($A < B \iff A \leq B$ and $B{-}A$ invertible)

$$[A, B] = \{C \in H_n \,:\, A \leq C \leq B\}$$

$$E_n = [0, I]$$

Bijective maps preserving order in both directions:

$$[A, B] \to [A + C, B + C]$$
$$X \mapsto X + C$$

$$[A, B] \to [TAT^*, TBT^*]$$
$$X \mapsto TXT^*$$

Bijective map satisfying $X \leq Y \iff \phi(Y) \leq \phi(X)$:

$0 < A < B$

$$[A, B] \to [B^{-1}, A^{-1}]$$
$$\phi(X) = X^{-1}$$

$$[0, I] \to [0, I]$$
$$\phi(X) = I - X$$

$$A \mapsto I + A \mapsto (I + A)^{-1} \mapsto$$

$$T(I+A)^{-1}T \mapsto I-T^2+T(I+A)^{-1}T \mapsto$$

$$(I - T^2 + T(I + A)^{-1}T)^{-1} \mapsto$$

$$(I - T^2 + T(I + A)^{-1}T)^{-1} - I$$

p a real number, $p < 1$.

$$f_p : [0,1] \to [0,1]$$

$$f_p(x) = \frac{x}{px + (1-p)}, \quad x \in [0,1].$$

THEOREM. $n \geq 2$. $\phi : E_n \to E_n$ bijective.

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\Downarrow$$

$\exists p, q \in (-\infty, 1)$, $\exists$ an invertible matrix T with $\|T\| \leq 1$ such that either

$$\phi(A) =$$

$$= f_q\left((f_p(TT^*))^{-1/2} \, f_p(TAT^*) \, (f_p(TT^*))^{-1/2} \right),$$

or

$$\phi(A) =$$

$$= f_q\left((f_p(TT^*))^{-1/2} \, f_p(TA^{tr}T^*) \, (f_p(TT^*))^{-1/2} \right).$$

Problem?

$A, B \in H_n$, $A < B$.

$$[A, B] = \{C \in H_n \,:\, A \leq C \leq B\},$$

$$[A, B) = \{C \in H_n \,:\, A \leq C < B\},$$

$$(A, B) = \{C \in H_n \,:\, A < C < B\}.$$

$$[A, \infty) = \{C \in H_n \,:\, C \geq A\},$$

$$(A, \infty) = \{C \in H_n \,:\, C > A\},$$

$$(-\infty, \infty) = H_n$$

$$(A, B],\ (-\infty, A],\ (-\infty, A)$$

Which of the above operator intervals are order isomorphic?

The general form of all order isomorphisms between operator intervals that are order isomorphic?

Simple reduction principle:

$I \sim J$ and $I_1 \sim J_1$ and we know isomorphisms. Then:

If we know the general form of all order isomorphisms between operator intervals I and I_1, then we know the general form of all order isomorphisms between operator intervals J and J_1.

Similar: $\sim$ denotes order anti-isomorphic

Each operator interval J is isomorphic to one of the following operator intervals:

$[0, I]$
$[0, \infty)$
$(-\infty, 0]$
$(0, \infty)$
$(-\infty, \infty)$

And any two of these operator intervals are order non-isomorphic.

The operator intervals $[0, \infty)$ and $(-\infty, 0]$ are obviously order anti-isomorphic. Hence, to understand the structure of all order isomorphisms between any two order isomorphic operator intervals it is enough to describe the general form of order automorphisms of the following four operator intervals:

$[0, I]$
$[0, \infty)$
$(0, \infty)$
$(-\infty, \infty)$

The group of order automorphisms of $[0, I]$ and $(-\infty, \infty)$: previous slides

THEOREM $\phi : [0, \infty) \to [0, \infty)$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exists an invertible matrix T such that either

$$\phi(A) = TAT^*$$

for every $A \in [0, \infty)$, or

$$\phi(A) = TA^{tr}T^*$$

for every $A \in [0, \infty)$.

THEOREM $\phi : (0,\infty) \to (0,\infty)$ a bijective map such that

$$A \leq B \iff \phi(A) \leq \phi(B).$$

Then there exists an invertible matrix T such that either

$$\phi(A) = TAT^*$$

for every $A \in (0,\infty)$, or

$$\phi(A) = TA^{tr}T^*$$

for every $A \in (0,\infty)$.

Optimality?

Can we replace the assumption

$$A \leq B \iff \phi(A) \leq \phi(B)$$

by the weaker one

$$A \leq B \Rightarrow \phi(A) \leq \phi(B)$$

and still get the same conclusion?

$$\phi : [0, \infty) \to [0, \infty)$$

$$\phi(A) = A^{1/2}$$

bijective map preserving order in one direction; operator monotone functions

Bijectivity? Essential in the infinite-dimensional case.

$$\begin{gathered} A, B \in H_n \ \text{ adjacent} \\ \Updownarrow \\ \operatorname{rank}(A - B) = 1 \end{gathered}$$

$\phi : H_n \to H_n$ preserves adjacency in both directions, if

$$A, B \ \text{adj} \iff \phi(A), \phi(B) \ \text{adj}$$

$$M = \{(x,y,z,t) \,:\, x,y,z,t \in \mathbf{R}\}$$

$$(x_1,y_1,z_1,t_1),(x_2,y_2,z_2,t_2) \in M \quad \text{coherent}$$
$$\Updownarrow$$
$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = c^2(t_1-t_2)^2$$

In mathematical foundations of relativity we usually use the harmless normalization $c = 1$.

Two space-time events are coherent (light-like) $\iff$ a light signal can be sent from one to the other

Alexandrov: description of bijective maps on M preserving coherency in both directions

$$r = (x,y,z,t) \leftrightarrow \begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix} = A$$

$$A \in H_2$$

$$\det A = t^2 - z^2 - x^2 - y^2$$

$$r_1, r_2 \in M, \quad r_j \leftrightarrow A_j$$

$$\begin{aligned} r_1, r_2 \ \text{coherent} \iff \det(A_2 - A_1) = 0 \\ \Updownarrow \\ A_2 - A_1 \ \text{singular} \\ \Updownarrow \\ A_1 = A_2 \ \text{or} \ A_1 \ \text{and} \ A_2 \ \text{adjacent} \end{aligned}$$

Thus, Alexandrov problem = study of adjacency preservers on H_2

$A, B \in H_n$, $A \neq B$. TFAE:

- A, B adj.
- A, B comparable and if C, D belong to operator interval between A and B, then C and D comparable.

Proof. ($\Downarrow$)
$B = A + tP$, say $t > 0 \Rightarrow A \leq B$

$$[A, B] = \{A + sP \,:\, 0 \leq s \leq t\}$$

$C, D \in [A, B] \Rightarrow C = A + s_1 P, D = A + s_2 P.$

($\Uparrow$) A, B not adjacent
If A, B not comparable, done.
If comparable, WLOG $A \leq B$. rank$(B - A) \geq 2 \Rightarrow$ "enough room" to find two noncomparable in $[A, B]$.