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## Banach limits and their applications

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1. Let $l_\infty$ be the space of all bounded sequence $x = (x_1, x_2, \ldots)$ with the norm $\|x\|_{l_\infty} = \sup\limits_{n\in\mathbb{N}} |x_n|$, and standard partial order. A linear functional $B \in l_\infty^*$ is said to be a Banach limit if $Bx \geqslant 0$ for any $x \in l_\infty$, $x \geqslant 0$, $B(1, 1, \ldots) = 1$ and $Bx = BTx$ for any $x \in \ell_\infty$ where $T$ is the translation operator.

The existence of Banach limits was established by S. Mazur and was presented in the famous S. Banach's book. Denote by $\mathfrak{B}$ the set of all Banach limits. It follows from the definition that $\mathfrak{B}$ is a convex closed set on the unit sphere of $l_\infty^*$ and that $\liminf\limits_{n\to\infty} x_n \leqslant Bx \leqslant \limsup\limits_{n\to\infty} x_n$ for any $B \in \mathfrak{B}$. Hence $Bx = \lim\limits_{n\to\infty} x_n$ for any convergent $x$.

G. Lorentz proved that for a given $x \in R^1$ and $x \in l_\infty$ the equality $Bx = a$ holds for every $B \in \mathfrak{B}$ iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k = a$$

uniformly in $m \in \mathbb{N}$. In this case we shall write $\operatorname*{Lim}_{n\to\infty} x_n = a$ and $x \in ac$.

For example:

1. $\operatorname{Lim}(-1)^n = 0.$
2. $\operatorname{Lim} \sin nt = 0 \ \forall\, t.$
3. $\operatorname{Lim} \sin^2 nt = \begin{cases} 0, & t = k\pi \\ {}^1\!/_2, & t \neq k\pi \end{cases}$
4. $\operatorname{Lim} |\sin nt| = \begin{cases} \frac{1}{q} \sum\limits_{k=1}^{q} \sin \frac{k\pi}{q}, & \frac{t}{\pi} = \frac{p}{q} \\ \frac{2}{\pi}, & \frac{t}{\pi} \text{ is not rational} \end{cases}$
5. $r_n(t) \overline{\in} ac$ for almost every $t \in [0, 1]$, where $\{r_n\}$ are the Rademacher functions.

L. Sucheston has sharpened the latter result by showing that for every $x \in l_\infty$

$$\{Bx : B \in \mathfrak{B}\} = [q(x), p(x)],$$

where

$$q(x) = \lim_{n\to\infty} \inf_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k, \quad p(x) = \lim_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k.$$

The diameter and radius of $\mathfrak{B}$ in $l_\infty^*$ are equal 2, i.e.

$$\sup_{B_1, B_2 \in \mathfrak{B}} \|B_1 - B_2\|_{l_\infty} = \inf_{B_1\in\mathfrak{B}} \sup_{B_2\in\mathfrak{B}} \|B_1 - B_2\|_{l_\infty^*} = 2.$$

2. We shall denote by $\Gamma$ the set of all operators $H$ satisfying the following conditions: $H \geqslant 0$, $H(1,1,\dots) = 1$, $Hc_0 \subset c_0$ and $\limsup_{j\to\infty}(A(I-T))_j \geqslant 0$ for all $x \in l_\infty$, $A \in R$, where $R = conv\{H^n, n = 1,2,\dots\}$. For any $H \in \Gamma$ there exists $B \in \mathfrak{B}$ s.t. $Bx = BHx$ for any $x \in l_\infty$. Denote by $\mathfrak{B}(H)$ the set of all Banach limits satisfying this condition. The Cesaro operator $C$ and the dilations operators $\sigma_n$ belong to $\Gamma$, where $(Cx)_n = \frac{1}{n}\sum_{k=1}^{n} x_k$,

$$\sigma_n(x_1, x_2, \dots) = (\underbrace{x_1, x_1, \dots, x_1}_{n}, \underbrace{x_2, x_2, \dots, x_2}_{n}, \dots).$$

The sets $\mathfrak{B}(\sigma_n)$, $n \in \mathbb{N}$ are distinct. The inclusion $\mathfrak{B}(\sigma_n) \subsetneqq \mathfrak{B}(\sigma_m)$ holds iff $m = n^2, n^3, \ldots$. Given $B \overline{\in} \mathfrak{B}(\sigma_m)$, there exists $B \in \mathfrak{B}(\sigma_n)$ s.t. $B \overline{\in} \mathfrak{B}(\sigma_m)\ \forall\, m \in \mathbb{N} \setminus \{n^2, n^3, \ldots\}$.

Note that $\bigcap\limits_{n=2}^{\infty} \mathfrak{B}(\sigma_n) = \bigcap\limits_{n=m}^{\infty} \mathfrak{B}(\sigma_m)$ for any $m \in \mathbb{N}$.

Let $m, i, j \in \mathbb{N}$, $m \geqslant 2$. The sets $\mathfrak{B}(\sigma_m)$ and $\mathfrak{B}(\sigma_{m^i}) \bigcap \mathfrak{B}(\sigma_{m^j})$ coinside iff $i, j$ are relatively prime.

Let $i_1, i_2, \ldots, i_n \in \mathbb{N}$, $i_1, i_2, \ldots, i_n \geqslant 2$. The set $\bigcap\limits_{k=1}^{n} \mathfrak{B}(\sigma_{i_k})$ coinsides with $\mathfrak{B}(\sigma_m)$ for some $m \in \mathbb{N}$ iff the numbers $\frac{\ln i_k}{\ln i_{k+1}}$ are rational for every $k = 1, 2, \ldots, n-1$.
For example, $\mathfrak{B}(\sigma_4) \bigcap \mathfrak{B}(\sigma_8) = \mathfrak{B}(\sigma_2)$ and $\mathfrak{B}(\sigma_2) \bigcap \mathfrak{B}(\sigma_3) \neq \mathfrak{B}(\sigma_m)$ for any $m \in \mathbb{N}$.

3. Every sequence $x \in l_\infty$ can be considered as a continuous founded function on the set $\mathbb{N}$ with the discrete topology. Therefore, this function extends uniquely to the continuous function $\widehat{x}$ from the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to $R^1$. This extension defines an isometric isomorphism from the space $l_\infty$ onto the space $C(\beta\mathbb{N})$.

Hence $l_\infty^*$ is in AL space and there exist a measurable space $\Omega$ with measure $\mu$ s.t. $\Omega = \Omega_d \bigcup \Omega_c$, $\mathfrak{B} = L_1(\Omega_d) \oplus L_1(\Omega_c)$ and the restrictions $\mu$ on $\Omega_d$ and $\Omega_c$ are discrete and continuous respectively. By Krein–Milman theorem

$$\mathfrak{B} = \overline{\mathrm{conv}}^w \, \mathrm{ext}\, \mathfrak{B}$$

where $\overline{\mathrm{conv}}^w \, \mathrm{ext}\, \mathfrak{B}$ is the closed convex hull of the set of extremal points of $\mathfrak{B}$ in weak$^*$ topology. Ch. Chow proved that the cardinality of $\mathrm{ext}\, \mathfrak{B}$ is $2^c$. Any countable set of $\mathrm{ext}\, \mathfrak{B}$ generates a subspace of $l_\infty^*$, which is isometric to $l_1$. Therefore $\|B_1 - B_2\|_{l_\infty^*} = 2$ for any $B_1, B_2 \in \mathrm{ext}\, \mathfrak{B}$, $B_1 \neq B_2$, i.e. $\mathfrak{B}$ may be interpreted as a simplex of dimension $2^c$ in $l_\infty$.

The set $\overline{\mathrm{conv}}^n \operatorname{ext} \mathfrak{B}$ where the closure is taken in the norm topology of $l_\infty^*$ is contained in $\mathfrak{B}$ and $\overline{\mathrm{conv}}^n \operatorname{ext} \mathfrak{B} \neq \mathfrak{B}$. Moreover $\mathfrak{B} \setminus \overline{\mathrm{conv}}^n \operatorname{ext} \mathfrak{B} \supset \mathfrak{B}(C)$. The set $\mathfrak{B}(C)$ is “large”: the cardinality of $\operatorname{ext} \mathfrak{B}(C)$ is $2^c$ and $\|B_1 - B_2\|_{l_\infty^*} = 2$ for any $B_1, B_2 \in \operatorname{ext} \mathfrak{B}(C)$.

4. There are two functional characteristics of a Banach limit. Given $B \in \mathfrak{B}$ and $t \in [0,1]$, denote

$$\varphi(B,t) = B\left(\bigcup_{n=1}^{\infty} [2^n, 2^{n+t})\right).$$

Clearly, $\varphi(B,0) = 0$, $\varphi(B,1) = 1$ and $\varphi(B,\cdot)$ increases on $[0,1]$. These properties characterize the function $\varphi$.

**Theorem**

*Let $f$ be an increasing function on $[0,1]$, $f(0)=0$, $f(1)=1$. There exists $B \in \mathfrak{B}$ s.t. $\varphi(B,t)=f(t)$ for every $t \in [0,1]$.*

If $B \in \operatorname{ext}\mathfrak{B}$, then $\varphi(B,t)=0$ or 1 for any $t \in [0,1]$. If $B \in \overline{\operatorname{conv}}^n \operatorname{ext}\mathfrak{B}$, then $\operatorname{mes}\{\varphi(B,t) : t \in [0,1]\} = 0$.
There exists $B \in \overline{\operatorname{conv}}^n \operatorname{ext}\mathfrak{B}$ s.t. $\{\varphi(B,t) : t \in [0,1]\} = K$ where $K$ is Cantor set.

Given $B \in \mathfrak{B}$, $x \in l_\infty$, $t \in R^1$, denote

$$F(B, x, t) = B\{n \in \mathbb{N}, x_n \leqslant t\}.$$

Clearly, $F(B, x, t)$ is an increasing function, $F(B, \mathbf{1}, t)$ is a discontinuous function.

**Theorem**

*The function $F(B, x, t)$ is continuous for some $x \in l_\infty$ iff $B \perp L_1(\Omega_d)$.*

For every $B \in \mathfrak{B}$ and for every increasing function $f$ s.t. $\min f(t) = 0$, $\max f(t) = 1$ there exists a sequence $x \in l_\infty$ s.t., $x_k = 0$ or $1$ for every $k \in \mathbb{N}$ and $F(B, x, t) = f(t)$ for each $t \in R^1$.

For every $B \in \mathfrak{B}$ and for every $s \in [0, 1]$ there exists a set $A \subset \mathbb{N}$ s.t. $BA = s$.

5. Let $W = \{w_n\}$ be the Walsh system and let

$$L_n(W) = \int\limits_0^1 |w_n(t)|\, dt, \quad n \in \mathbb{N}$$

be the Lebesque constants of the Walsh system. It is known that the sequence $\frac{1}{\log n} L_n(W)$, $n \geqslant 2$ is bounded.

## Theorem

*1.* $0 \leqslant B\left(\frac{1}{\log_2 n}L_n(W)\right) \leqslant \frac{1}{3}$ *for any* $B \in \mathfrak{B}$.

*2. If* $B \in \mathfrak{B}(C)$, *then* $B\left(\frac{1}{\log_2 n}L_n(W)\right) = \frac{1}{4}$.

The constants 0 and $\frac{1}{3}$ in Th. 3 are exact. Therefore the sequence $\left\{\frac{1}{\log_2 n}L_n(W)\right\}$ does not belong to the space *ac*.
This result was proved jointly with S. Astashkin.

Let $1 < p < \infty$ and $L_{p,\infty}$ be the Marcinkiewich space endowed with the norm

$$\|x\|_{L_{p,\infty}} = \sup_{s \subset [0,1], \mathrm{mes}\, e > 0} (\mathrm{mes}\, e)^{\frac{1}{p}-1} \int\limits_{e} |x(t)| dt.$$

Consider the operator

$$Ax(s) = B\left(2^{n\left(1-\frac{1}{p}\right)} \int\limits_{s}^{s+2^{-n}} x(t)\, dt\right)$$

where $B \in \mathfrak{B}$, $s \in (0,1)$.

### Theorem

*1. The operator $A$ acts from $L_{p,\infty}$ into $l_p(0,1)$ and its norm is equal* $1$.
*2. If $y \in l_p(0,1)$, then there exists $x \in L_{p,\infty}$ s.t. $Ax = y$ and* $\|x\|_{L_{p,\infty}} = \|y\|_{l_p(0,1)}$.

This result was proved jointly with A. Shteinberg.

Let $B(H)$ be the algebra of bounded linear operators on a separable Hilbert space $H$. For every $A \in B(H)$ denote by $\lambda_n(A)$ the $n$-th eigenvalue of the operator $A$. Any $B \in \mathfrak{B}$ generates so called singular trace $\tau_B$ on $B(H)$ given by the formula

$$\tau_B(A) = \begin{cases} B(\sum_{2^n-1}^{2^{n+1}-2} \lambda_n(A)), \text{if the sums are uniformly bounded,} \\ \infty, \text{otherwise.} \end{cases}$$

The construction of singular traces originated in the J. Dixmier work in 1966. Since than it was developed in many different direction (see for example the monograph S. Lord, F. Sukochev and A. Zanin. Singular Thaces. Theory and Applications. Studies in Mathematics. Vol. 46, De Gruyter 2012). Recently it was established that the mapping $B \to \tau_B$ defines an isometry between the set of Banach limits and that of positive normalised singular traces.

Final example:

If $a > 1$, $x = (x_1, x_2, \ldots)$ and $x_k = (-1)^n$ for $a^n \leqslant k < a^{n+1}$, $k, n \in \mathbb{N}$, then

1. $\{Bx : B \in \mathfrak{B}\} = [-1, 1]$,
2. $\{Bx : B \in \operatorname{ext} \mathfrak{B}\} = \{-1, 1\}$,
3. $\{Bx : B \in \mathfrak{B}(C)\} = \{0\}$.