Riesz-Kantorovich formulas for operators on multi-wedged spaces

Christopher M. Schwanke
Department of Mathematics
North-West University

July 20, 2017
Positivity IX
University of Alberta

Joint work with Marten Wortel
In recent papers by Marcel de Jeu and Miek Messerschmidt, we see a new direction on ordered and preordered vector spaces emerge which involves vector spaces equipped with an arbitrary set of wedges.
In recent papers by Marcel de Jeu and Miek Messerschmidt, we see a new direction on ordered and preordered vector spaces emerge which involves vector spaces equipped with an arbitrary set of wedges.

**Definition**

For a vector space $E$, we call a nonempty subset $W$ of $E$ a wedge if $W + W \subseteq W$ and $\lambda W \subseteq W$ for all $0 \leq \lambda \in \mathbb{R}$.
In recent papers by Marcel de Jeu and Miek Messerschmidt, we see a new direction on ordered and preordered vector spaces emerge which involves vector spaces equipped with an arbitrary set of wedges.

**Definition**
For a vector space $E$, we call a nonempty subset $W$ of $E$ a wedge if $W + W \subseteq W$ and $\lambda W \subseteq W$ for all $0 \leq \lambda \in \mathbb{R}$. In this case, $(E, W)$ is called a preordered vector space.
In recent papers by Marcel de Jeu and Miek Messerschmidt, we see a new direction on ordered and preordered vector spaces emerge which involves vector spaces equipped with an arbitrary set of wedges.

**Definition**

For a vector space $E$, we call a nonempty subset $W$ of $E$ a wedge if $W + W \subseteq W$ and $\lambda W \subseteq W$ for all $0 \leq \lambda \in \mathbb{R}$. In this case, $(E, W)$ is called a preordered vector space. A cone $K$ is a wedge that satisfies $K \cap (-K) = \{0\}$.
In recent papers by Marcel de Jeu and Miek Messerschmidt, we see a new direction on ordered and preordered vector spaces emerge which involves vector spaces equipped with an arbitrary set of wedges.

**Definition**

For a vector space $E$, we call a nonempty subset $W$ of $E$ a wedge if $W + W \subseteq W$ and $\lambda W \subseteq W$ for all $0 \leq \lambda \in \mathbb{R}$. In this case, $(E, W)$ is called a preordered vector space. A cone $K$ is a wedge that satisfies $K \cap (-K) = \{0\}$. In this case $(E, K)$ is called an ordered vector space.
Multi-wedged spaces

Definition

We call a pair \((E, \mathcal{W})\) a multi-wedged space if \(E\) is a vector space and \(\mathcal{W}\) is a nonempty set of wedges in \(E\).
Multi-wedged spaces

Definition

We call a pair \((E, \mathcal{W})\) a multi-wedged space if \(E\) is a vector space and \(\mathcal{W}\) is a nonempty set of wedges in \(E\).

The idea in the aforementioned work of Marcel de Jeu and Miek Messerschmidt was to extend some classical results for ordered vector spaces to results that hold for special types of multi-wedged spaces.
Theorem (Andô’s theorem)

Let $E$ be a real Banach space ordered by a closed cone $K$ for which $E = K - K$. 
Theorem (Andô’s theorem)

Let $E$ be a real Banach space ordered by a closed cone $K$ for which $E = K - K$. Then there exists a constant $C > 0$ such that for every $x \in E$ there exist $y \in K$ and $z \in -K$ for which $x = y + z$ and $\|y\| + \|z\| \leq C\|x\|$.
Theorem (de Jeu, Messerschmidt)

Let \((E, \mathcal{W})\) be a multi-wedged space, where \(E\) is a Banach space, and let \(\{W_i\}_{i \in I}\) be a collection of closed wedges in \(\mathcal{W}\) for which every \(x \in E\) can be written as an absolutely convergent series
\[
x = \sum_{i \in I} w_i, \text{ with } w_i \in W_i.
\]
Andô’s theorem extended

**Theorem (de Jeu, Messerschmidt)**

Let \((E, \mathcal{W})\) be a multi-wedged space, where \(E\) is a Banach space, and let \(\{W_i\}_{i \in I}\) be a collection of closed wedges in \(\mathcal{W}\) for which every \(x \in E\) can be written as an absolutely convergent series
\[ x = \sum_{i \in I} w_i, \text{ with } w_i \in W_i. \]

Then there exist continuous positively homogeneous maps \(\gamma_i : E \to W_i\) such that
Theorem (de Jeu, Messerschmidt)

Let \((E, \mathcal{W})\) be a multi-wedged space, where \(E\) is a Banach space, and let \(\{W_i\}_{i \in I}\) be a collection of closed wedges in \(\mathcal{W}\) for which every \(x \in E\) can be written as an absolutely convergent series \(x = \sum_{i \in I} w_i\), with \(w_i \in W_i\).

Then there exist continuous positively homogeneous maps \(\gamma_i : E \to W_i\) such that

\[
(1.)\ x = \sum_{i \in I} \gamma_i(x)\quad \text{for all } x \in E,
\]
Theorem (de Jeu, Messerschmidt)

Let \((E, \mathcal{W})\) be a multi-wedged space, where \(E\) is a Banach space, and let \(\{W_i\}_{i \in I}\) be a collection of closed wedges in \(\mathcal{W}\) for which every \(x \in E\) can be written as an absolutely convergent series
\[ x = \sum_{i \in I} w_i, \text{ with } w_i \in W_i. \]

Then there exist continuous positively homogeneous maps \(\gamma_i : E \to W_i\) such that

1. \(x = \sum_{i \in I} \gamma_i(x)\) for all \(x \in E\),
2. \(\sum_{i \in I} \|\gamma_i(x)\| \leq C\|x\|\) for all \(x \in E\).
A curious mind who is interested in vector lattices and multi-wedged spaces could very well ask if results from vector lattice theory can likewise be extended to certain multi-wedged spaces.
In this talk, we’ll focus on extending the Riesz-Kantorovich formulas to the multi-wedged setting.
Theorem (Riesz-Kantorovich formulas)

Suppose \((E, W)\) is a preordered vector space with the Riesz decomposition property, and assume \(E = W - W\).

Note the importance of the RDP.

Christopher M. Schwanke

Riesz-Kantorovich formulas for multi-wedged spaces
The Riesz-Kantorovich formulas

Theorem (Riesz-Kantorovich formulas)

Suppose \((E, W)\) is a preordered vector space with the Riesz decomposition property, and assume \(E = W - W\). Let \((F, F^+)\) be a Dedekind complete vector lattice.

Note the importance of the RDP.
Theorem (Riesz-Kantorovich formulas)

Suppose \((E, W)\) is a preordered vector space with the Riesz decomposition property, and assume \(E = W - W\). Let \((F, F^+)\) be a Dedekind complete vector lattice. Then \((\mathcal{L}_b(E, F), \mathcal{L}_b^+(E, F))\) is a Dedekind complete vector lattice.
Suppose \((E, W)\) is a preordered vector space with the Riesz decomposition property, and assume \(E = W - W\). Let \((F, F^+)\) be a Dedekind complete vector lattice. Then \((\mathcal{L}_b(E, F), \mathcal{L}_b^+(E, F))\) is a Dedekind complete vector lattice. For \(T_1, T_2 \in \mathcal{L}_b(E, F)\) and \(x \in W\),

\[
(T_1 \vee T_2)(x) = \sup \{ T_1(y_1) + T_2(y_2) : y_1, y_2 \in W, y_1 + y_2 = x \}.
\]
Theorem (Riesz-Kantorovich formulas)

Suppose \((E, W)\) is a preordered vector space with the Riesz decomposition property, and assume \(E = W - W\). Let \((F, F^+)\) be a Dedekind complete vector lattice. Then \((\mathcal{L}_b(E, F), \mathcal{L}_b^+(E, F))\) is a Dedekind complete vector lattice. For \(T_1, T_2 \in \mathcal{L}_b(E, F)\) and \(x \in W\),

\[
(T_1 \vee T_2)(x) = \sup \{ T_1(y_1) + T_2(y_2) : y_1, y_2 \in W, y_1 + y_2 = x \}.
\]

Note the importance of the RDP.
Our first step in obtaining multi-wedged Riesz-Kantorovich formulas is to generalize the concept of suprema in ordered vector spaces to the multi-wedged setting.
Remark

For an ordered vector space \((E, K)\) and a collection \((x_i)_{i \in I}\) in \(E\),
Remark

For an ordered vector space \((E, K)\) and a collection \((x_i)_{i \in I}\) in \(E\), it is true that \(z = \sup_{i \in I} \{x_i\}\) if and only if \(\bigcap_{i \in I} (x_i + K) = z + K\).
A geometrical interpretation of suprema

Remark

For an ordered vector space $(E, K)$ and a collection $(x_i)_{i \in I}$ in $E$, it is true that $z = \sup_{i \in I} \{x_i\}$ if and only if $\bigcap_{i \in I} (x_i + K) = z + K$. 

Christopher M. Schwanke

Riesz-Kantorovich formulas for multi-wedged spaces
A geometrical interpretation of suprema
A geometrical interpretation of suprema
Remark

If \((E, \mathcal{W})\) is a multi-wedged space and \((x_i, W_i)_{i \in I}\) is a collection in \(E \times \mathcal{W}\) then any \(z \in E\) that satisfies

\[
\bigcap_{i \in I} (x_i + W_i) = z + \bigcap_{i \in I} W_i
\]

can be viewed as a generalized supremum of \((x_i, W_i)_{i \in I}\).
Generalized suprema

Remark

If \((E, \mathcal{W})\) is a multi-wedged space and \((x_i, W_i)_{i \in I}\) is a collection in \(E \times \mathcal{W}\) then any \(z \in E\) that satisfies

\[
\bigcap_{i \in I} (x_i + W_i) = z + \bigcap_{i \in I} W_i
\]
Remark

If \((E, \mathcal{W})\) is a multi-wedged space and \((x_i, W_i)_{i \in I}\) is a collection in \(E \times \mathcal{W}\) then any \(z \in E\) that satisfies

\[
\bigcap_{i \in I} (x_i + W_i) = z + \bigcap_{i \in I} W_i
\]

can be viewed as a generalized supremum of \((x_i, W_i)_{i \in I}\).
Generalized suprema

Riesz-Kantorovich formulas for multi-wedged spaces
Generalized suprema

Riesz-Kantorovich formulas for multi-wedged spaces
Definition

We refer to a generalized suprema of $(x_i, W_i)_{i \in I}$ as a multi-suprema of $(x_i, W_i)_{i \in I}$.
Definition

We refer to a generalized suprema of \((x_i, W_i)_{i \in I}\) as a multi-suprema of \((x_i, W_i)_{i \in I}\). The set of all multi-suprema of \((x_i, W_i)_{i \in I}\) is denoted \(\text{msup}(x_i, W_i)_{i \in I}\).
Multi-suprema

**Definition**

We refer to a generalized suprema of \((x_i, W_i)_{i \in I}\) as a multi-suprema of \((x_i, W_i)_{i \in I}\). The set of all multi-suprema of \((x_i, W_i)_{i \in I}\) is denoted \(\text{msup}(x_i, W_i)_{i \in I}\).

**Remark**

*In order for such a set of multi-suprema to be nonempty, \((x_i, W_i)_{i \in I}\) must be multi-bounded above,*
Multi-suprema

**Definition**

We refer to a generalized suprema of \((x_i, W_i)_{i \in I}\) as a multi-suprema of \((x_i, W_i)_{i \in I}\). The set of all multi-suprema of \((x_i, W_i)_{i \in I}\) is denoted \(\text{msup}_{i \in I}(x_i, W_i)\).

**Remark**

*In order for such a set of multi-suprema to be nonempty, \((x_i, W_i)_{i \in I}\) must be multi-bounded above, meaning that \(\bigcap_{i \in I}(x_i + W_i) \neq \emptyset\).*
Generalized vector lattices

Definition

Multi-wedged spaces in which \( \text{msup}(x_i, W_i) \neq \emptyset \) for all \( i \in I \) multi-bounded above collections \( (x_i, W_i)_{i \in I} \) with \( |I| \leq \kappa \) are called \( \kappa \)-multi-lattices.

Christopher M. Schwanke

Riesz-Kantorovich formulas for multi-wedged spaces
Generalized vector lattices

**Definition**
Multi-wedged spaces in which $\text{msup}(x_i, W_i) \neq \emptyset$ for all $i \in I$ multi-bounded above collections $(x_i, W_i)_{i \in I}$ with $|I| \leq \kappa$ are called $\kappa$-multi-lattices.

**Definition**
Dedekind complete multi-lattices are multi-wedged spaces that are $\kappa$-multi-lattices for any cardinal number $\kappa$. 

Christopher M. Schwanke
Riesz-Kantorovich formulas for multi-wedged spaces
Consider the vector space $E = \mathbb{R}^{[0,2]}$. 

Example 1
Consider the vector space $E = \mathbb{R}^{[0,2]}$. Define

$W_{[0,1]} = \{ f \in E: f(x) \geq 0 \text{ for all } x \in [0, 1] \},$
Example 1

Consider the vector space $E = \mathbb{R}^{[0,2]}$. Define

$W_{[0,1]} = \{ f \in E : f(x) \geq 0 \text{ for all } x \in [0,1] \}$, and

$W_{(1,2]} = \{ f \in E : f(x) \geq 0 \text{ for all } x \in (1,2] \}$. 
Find $\text{msup}( (f_1, W_{[0,1]}), (f_2, W_{[0,1]}), (f_3, W_{(1,2)}), (f_4, W_{(1,2)}) )$. 
Example 1 continued

\[ m_{\sup}(f_1, W_{[0,1]}), (f_2, W_{[0,1]}), (f_3, W_{(1,2)}), (f_4, W_{(1,2)}) \]
Remark

We can infer from this example that \((\mathbb{R}^{[0,2]}, \{W_{[0,1]}, W_{(1,2]}\})\) is a Dedekind complete multi-lattice.
Example 1 continued

Remark

We can infer from this example that \((\mathbb{R}^{[0,2]}, \{W_{[0,1]}, W_{(1,2)}\})\) is a Dedekind complete multi-lattice.

Remark

We also see that the particular multi-supremum in this example is unique.
Remark

**We can infer from this example that** \((\mathbb{R}^{[0,2]}, \{W_{[0,1]}, W_{(1,2)}\})\) is a Dedekind complete multi-lattice.

Remark

We also see that the particular multi-supremum in this example is unique.

Remark

\(\text{msup}(x_i, W_i)\) is a singleton set if and only if \(\bigcap_{i \in I} W_i\) is a cone.
“Lost in Abstraction”

*Besides losing uniqueness of suprema in the venturing from vector lattices to multi-lattices,*
“Lost in Abstraction”

Besides losing uniqueness of suprema in the venturing from vector lattices to multi-lattices, we also lose the inductive property that vector lattices are closed under finite suprema.
“Lost in Abstraction”

Besides losing uniqueness of suprema in the venturing from vector lattices to multi-lattices, we also lose the inductive property that vector lattices are closed under finite suprema. Indeed, there exist multi-wedged spaces that are \( n \)-multi-lattices but not \((n + 1)\)-multi-lattices.
Proper multi-suprema

Remark

*It is of particular interest when the set of multi-suprema is a singleton set.*
Remark

It is of particular interest when the set of multi-suprema is a singleton set. In this case we say we have a proper multi-supremum.
Remark

*It is of particular interest when the set of multi-suprema is a singleton set. In this case we say we have a proper multi-supremum. For sake of time, we’ll only focus on proper multi-suprema from now on.*
Definition
Let \((E, \mathcal{W})\) and \((F, \mathcal{V})\) be multi-wedged spaces.
**Definition**

Let \((E, \mathcal{W})\) and \((F, \mathcal{V})\) be multi-wedged spaces. For \(W \in \mathcal{W}\) and \(V \in \mathcal{V}\), we say that a map \(T: E \to F\) is \((W, V)\)-positive if \(T(W) \subseteq V\).
Definition

Let \((E, \mathcal{W})\) and \((F, \mathcal{V})\) be multi-wedged spaces. For \(W \in \mathcal{W}\) and \(V \in \mathcal{V}\), we say that a map \(T : E \to F\) is \((W, V)\)-positive if \(T(W) \subseteq V\). We denote by \(\mathcal{L}_{W, V}(E, F)\) the set of all \((W, V)\)-positive operators \(T : E \to F\).
Definition

Let \((E, \mathcal{W})\) and \((F, \mathcal{V})\) be multi-wedged spaces. For \(W \in \mathcal{W}\) and \(V \in \mathcal{V}\), we say that a map \(T : E \to F\) is \((W, V)\)-positive if \(T(W) \subseteq V\). We denote by \(\mathcal{L}_{W, V}(E, F)\) the set of all \((W, V)\)-positive operators \(T : E \to F\). Also, we set

\[ \mathcal{L}_{W, V}(E, F) = \{ \mathcal{L}_{W, V}(E, F) : W \in \mathcal{W}, V \in \mathcal{V} \} \]
Definition
Let \((E, \mathcal{W})\) and \((F, \mathcal{V})\) be multi-wedged spaces. For \(W \in \mathcal{W}\) and \(V \in \mathcal{V}\), we say that a map \(T: E \to F\) is \((W, V)\)-positive if \(T(W) \subseteq V\). We denote by \(L_{W,V}(E, F)\) the set of all \((W, V)\)-positive operators \(T: E \to F\). Also, we set

\[L_{W,V}(E, F) = \{L_{W,V}(E, F) : W \in \mathcal{W}, V \in \mathcal{V}\}\].

Proposition
\((L(E, F), L_{W,V}(E, F))\) is a multi-wedged space.
Since we wish to obtain Riesz-Kantorovich formulas for multi-wedged spaces of operators, we need a natural generalization of the Riesz decomposition property for the multi-wedged setting.
Riesz decomposition property

Definition

\((E, \mathcal{W})\) has the \((m, n)\)-Riesz decomposition property if for any \(W_1, \ldots, W_n \in \mathcal{W}\) and any \(x_1, \ldots, x_m \in \sum_{j=1}^{n} W_j\) and \(y_1 \in W_1, \ldots, y_n \in W_n\) such that \(m \sum_{i=1}^{m} x_i = n \sum_{j=1}^{n} y_j\), there exist \(z_{ij} \in W_j\) for which \(x_i = n \sum_{j=1}^{n} z_{ij}\) and \(y_j = m \sum_{i=1}^{m} z_{ij}\).
Definition

\((E, \mathcal{W})\) has the \((m, n)\)-Riesz decomposition property if for any \(W_1, \ldots, W_n \in \mathcal{W}\) and any \(x_1, \ldots, x_m \in \sum_{j=1}^{n} W_j\) and any \(y_1 \in W_1, \ldots, y_n \in W_n\) such that

\[
\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j,
\]

there exist \(z_{ij} \in W_j\) for which

\[
x_i = \sum_{j=1}^{n} z_{ij}, \quad y_j = \sum_{i=1}^{m} z_{ij}.
\]
Riesz decomposition property

Definition

$(E, \mathcal{W})$ has the $(m, n)$-Riesz decomposition property if for any $W_1, \ldots, W_n \in \mathcal{W}$ and any $x_1, \ldots, x_m \in \sum_{j=1}^{n} W_j$ and $y_1 \in W_1, \ldots, y_n \in W_n$ there exist $z_{ij} \in W_j$ for which $x_i = \sum_{j=1}^{n} z_{ij}$ and $y_j = \sum_{i=1}^{m} z_{ij}$. 
Riesz decomposition property

Definition

$(E, \mathcal{W})$ has the $(m, n)$-Riesz decomposition property if for any $W_1, \ldots, W_n \in \mathcal{W}$ and any $x_1, \ldots, x_m \in \sum_{j=1}^{n} W_j$ and $y_1 \in W_1, \ldots, y_n \in W_n$ such that

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j,$$
Riesz decomposition property

**Definition**

$(E, \mathcal{W})$ has the $(m, n)$-Riesz decomposition property if for any $W_1, \ldots, W_n \in \mathcal{W}$ and any $x_1, \ldots, x_m \in \sum_{j=1}^{n} W_j$ and $y_1 \in W_1, \ldots, y_n \in W_n$ such that

$$
\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j,
$$

there exist $z_{ij} \in W_j$ for which
Riesz decomposition property

**Definition**

$(E, \mathcal{W})$ has the $(m, n)$-Riesz decomposition property if for any $W_1, \ldots, W_n \in \mathcal{W}$ and any $x_1, \ldots, x_m \in \sum_{j=1}^n W_j$ and $y_1 \in W_1, \ldots, y_n \in W_n$ such that

$$
\sum_{i=1}^m x_i = \sum_{j=1}^n y_j,
$$

there exist $z_{ij} \in W_j$ for which

$$
x_i = \sum_{j=1}^n z_{ij} \quad \text{and} \quad y_j = \sum_{i=1}^m z_{ij}.
$$
There exist (Dedekind complete) multi-lattices that do not even have the $(2, 2)$-RDP.
“Lost in Abstraction”

There exist (Dedekind complete) multi-lattices that do not even have the $(2, 2)$-RDP.

“Lost in Abstraction”

There exist multi-wedged spaces that have the $(m, n)$-RDP but not the $(m, n + 1)$-RDP.
Theorem

Let \((E, \mathcal{W})\) be a multi-wedged space
Main theorem

**Theorem**

Let \((E, \mathcal{W})\) be a multi-wedged space and \((F, V)\) be an ordered vector space that is a Dedekind complete multi-lattice.
Let \((E, \mathcal{W})\) be a multi-wedged space and \((F, V)\) be an ordered vector space that is a Dedekind complete multi-lattice. Consider the multi-wedged space \((\mathcal{L}(E, F), \mathcal{L}_{\mathcal{W}, \{V\}}(E, F))\).
Main theorem

**Theorem**

Let \((E, \mathcal{W})\) be a multi-wedged space and \((F, V)\) be an ordered vector space that is a Dedekind complete multi-lattice. Consider the multi-wedged space \((\mathcal{L}(E, F), \mathcal{L}_{\mathcal{W},\mathcal{V}}(E, F))\). Also consider a multi-bounded above collection \((T_i, \mathcal{L}_{W_i,V}(E, F))_{i \in I}\).
Main theorem

**Theorem**

Let $(E, \mathcal{W})$ be a multi-wedged space and $(F, V)$ be an ordered vector space that is a Dedekind complete multi-lattice. Consider the multi-wedged space $(\mathcal{L}(E, F), \mathcal{L}_{\mathcal{W}, \{V\}}(E, F))$. Also consider a multi-bounded above collection $(T_i, \mathcal{L}_{W_i, V}(E, F))_{i \in I}$. Assume $E = \sum_{i \in I} W_i - \sum_{i \in I} W_i$. 

Christopher M. Schwanke

Riesz-Kantorovich formulas for multi-wedged spaces
Main theorem continued

**Theorem (continued)**

*If either*

1. $|I| \leq n$ and $(E, \mathcal{W})$ has the $(2, n)$-RDP,

2. the cardinality of $I$ is arbitrary and $(E, \mathcal{W})$ has the $(2, n)$-RDP for every $n \in \mathbb{N}$

*then for $x \in \sum_{i \in I} W_i$,*

$$\sup_{i \in I} \left( T_i \left( \sum_{i \in I} W_i \right), V(E, F) \right)(x) = \sup \{ \sum_{i \in I} T_i(y_i) : (y_i)_{i \in I} \in \bigoplus_{i \in I} W_i, \sum_{i \in I} y_i = x \}.$$
Theorem (continued)

If either

1. $|I| \leq n$ and $(E, \mathcal{W})$ has the $(2, n)$-RDP, or
2. the cardinality of $I$ is arbitrary and $(E, \mathcal{W})$ has the $(2, n)$-RDP for every $n \in \mathbb{N}$
Main theorem continued

Theorem (continued)

If either

(1) $|I| \leq n$ and $(E, \mathcal{W})$ has the $(2, n)$-RDP, or

(2) the cardinality of $I$ is arbitrary and $(E, \mathcal{W})$ has the $(2, n)$-RDP for every $n \in \mathbb{N}$

then for $x \in \sum_{i \in I} W_i$. 
Theorem (continued)

If either

(1) \( |I| \leq n \) and \((E, \mathcal{W})\) has the \((2, n)\)-RDP, or

(2) the cardinality of \( I \) is arbitrary and \((E, \mathcal{W})\) has the \((2, n)\)-RDP for every \( n \in \mathbb{N} \)

then for \( x \in \sum_{i \in I} W_i \),

\[
\text{msup}_{i \in I} (T_i, \mathcal{L}_{W_i}, \nu(E, F))(x) = \\
\sup \left\{ \sum_{i \in I} T_i(y_i) : (y_i)_{i \in I} \in \bigoplus_{i \in I} W_i, \sum_{i \in I} y_i = x \right\}.
\]
Theorem (continued)

In particular, under the assumptions of (1) we have that
\((\mathcal{L}(E, F), \mathcal{L}_W, \{V\}(E, F))\) is an \(n\)-multi-lattice, whereas
\((\mathcal{L}(E, F), \mathcal{L}_W, \{V\}(E, F))\) is a Dedekind complete multi-lattice
under the assumptions of (2).

(1) \(|I| \leq n\) and \((E, W)\) has the \((2, n)\)-RDP,
(2) the cardinality of \(I\) is arbitrary and \((E, W)\) has the \((2, n)\)-RDP
for every \(n \in \mathbb{N}\)
A more general case

Remark

This theorem is also valid even if $E \neq \sum_{i \in I} W_i - \sum_{i \in I} W_i$ and when $V$ is a wedge that is not a cone, but the Riesz-Kantorovich formulas get a bit unwieldy.
Acknowledgement

This research was partially funded by the Claude Leon Foundation and by the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS).
Thank you for listening!