Riesz-Kantorovich formulas for operators on multi-wedged spaces

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July 20, 2017
Positivity IX
University of Alberta

Joint work with Marten Wortel

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The idea in the aforementioned work of Marcel de Jeu and Miek Messerschmidt was to extend some classical results for ordered vector spaces to results that hold for special types of multi-wedged spaces.

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Let E be a real Banach space ordered by a closed cone K for which $E = K - K$. Then there exists a constant $C > 0$ such that for every $x \in E$ there exist $y \in K$ and $z \in -K$ for which $x = y + z$ and $||y|| + ||z|| \leq C||x||$.

Andô's theorem extended

Theorem (de Jeu, Messerschmidt)

Let $(E, \mathcal{W})$ be a multi-wedged space, where E is a Banach space, and let $\{W_i\}_{i \in I}$ be a collection of closed wedges in $\mathcal{W}$ for which every $x \in E$ can be written as an absolutely convergent series $x = \sum_{i \in I} w_i$, with $w_i \in W_i$.

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Then there exist continuous positively homogeneous maps $\gamma_i : E \to W_i$ such that

(1.) $x = \sum_{i\in I} \gamma_i(x)$ *for all* $x \in E$,

(2.) $\sum_{i\in I} ||\gamma_i(x)|| \leq C||x||$ *for all* $x \in E$.

Multi-wedged vector lattices?

A curious mind who is interested in vector lattices and multi-wedged spaces could very well ask if results from vector lattice theory can likewise be extended to certain multi-wedged spaces.

The Riesz-Kantorovich formulas

In this talk, we'll focus on extending the Riesz-Kantorovich formulas to the multi-wedged setting.

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Theorem (Riesz-Kantorovich formulas)

Suppose (E, W) is a preordered vector space with the Riesz decomposition property, and assume $E = W - W$. Let (F, F^+) be a Dedekind complete vector lattice. Then $(\mathcal{L}_b(E, F), \mathcal{L}_b^+(E, F))$ is a Dedekind complete vector lattice.

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Theorem (Riesz-Kantorovich formulas)

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$$(T_1 \vee T_2)(x) = \sup\{T_1(y_1) + T_2(y_2) \colon y_1, y_2 \in W, y_1 + y_2 = x\}.$$

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$$\big(T_1 \vee T_2\big)(x) = \sup\left\{T_1(y_1) + T_2(y_2) \colon y_1, y_2 \in W, y_1 + y_2 = x\right\}.$$

Note the importance of the RDP.

Our first step in obtaining multi-wedged Riesz-Kantorovich formulas is to generalize the concept of suprema in ordered vector spaces to the multi-wedged setting.

A geometrical interpretation of suprema

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A geometrical interpretation of suprema

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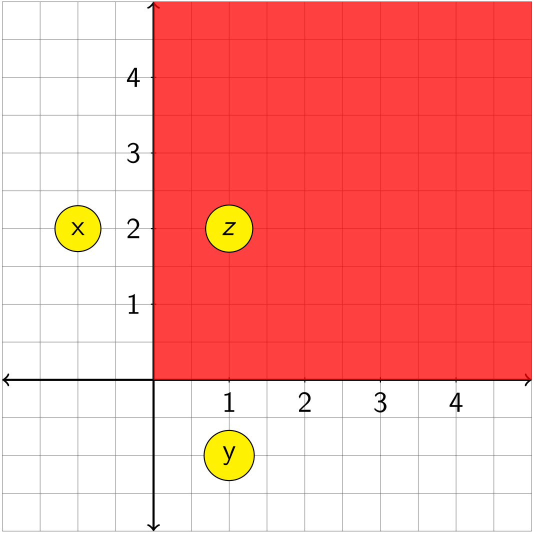
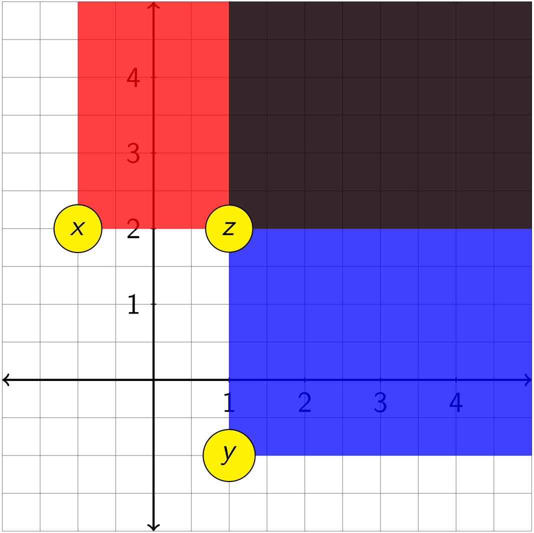
For an ordered vector space (E, K) and a collection $(x_i)_{i\in I}$ in E, it is true that $z = \sup_{i\in I}\{x_i\}$ if and only if $\bigcap_{i\in I}(x_i + K) = z + K$.

A geometrical interpretation of suprema



A geometrical interpretation of suprema



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If $(E, \mathcal{W})$ *is a multi-wedged space and* $(x_i, W_i)_{i \in I}$ *is a collection in* $E \times \mathcal{W}$

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$$\bigcap_{i \in I} (x_i + W_i) = z + \bigcap_{i \in I} W_i$$

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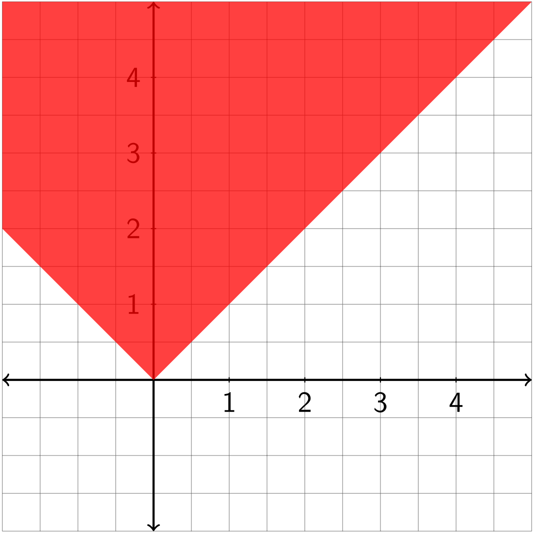
If $(E, \mathcal{W})$ is a multi-wedged space and $(x_i, W_i)_{i\in I}$ is a collection in $E \times \mathcal{W}$ then any $z \in E$ that satisfies

$$\bigcap_{i\in I}(x_i + W_i) = z + \bigcap_{i\in I} W_i$$

can be viewed as a generalized supremum of $(x_i, W_i)_{i\in I}$.

Generalized suprema



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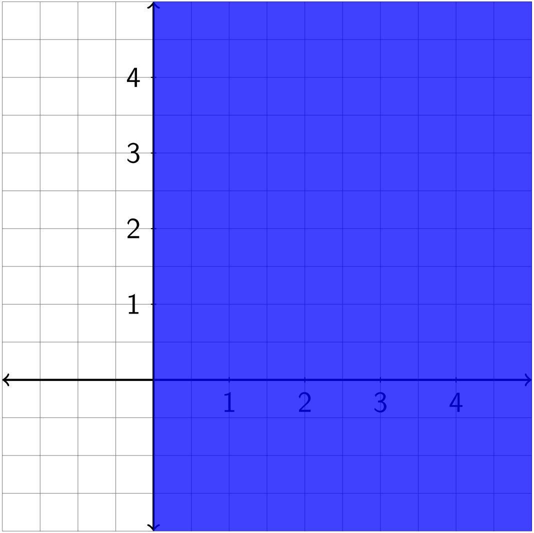

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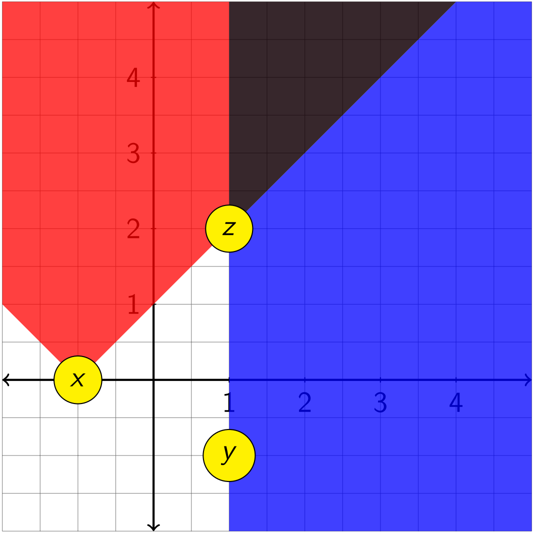

Multi-suprema

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Remark

In order for such a set of multi-suprema to be nonempty, $(x_i, W_i)_{i\in I}$ must be multi-bounded above, meaning that $\bigcap_{i\in I}(x_i + W_i) \neq \varnothing$.

Generalized vector lattices

Definition

Multi-wedged spaces in which $\operatorname*{msup}_{i\in I}(x_i, W_i) \neq \varnothing$ for all multi-bounded above collections $(x_i, W_i)_{i\in I}$ with $|I| \leq \kappa$ are called κ-multi-lattices.

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Definition

Dedekind complete multi-lattices are multi-wedged spaces that are κ-multi-lattices for any cardinal number κ.

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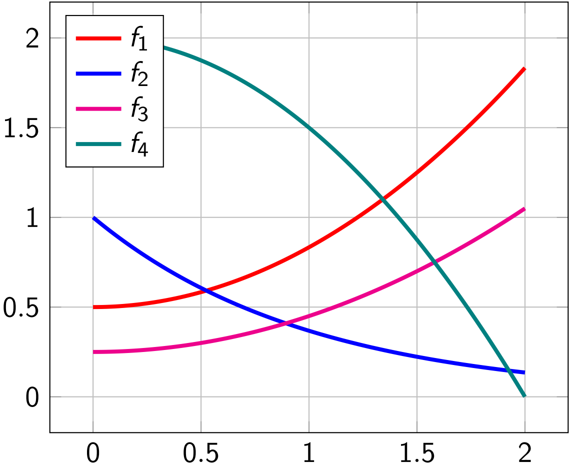
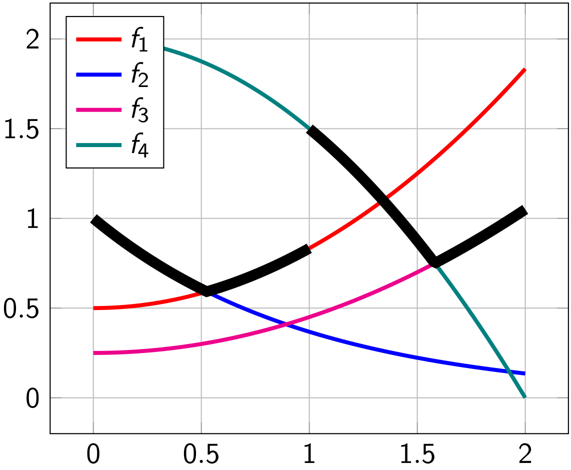
$W_{[0,1]} = \{f \in E \colon f(x) \geq 0 \text{ for all } x \in [0,1]\}$, *and*

$W_{(1,2]} = \{f \in E \colon f(x) \geq 0 \text{ for all } x \in (1,2]\}$.



Find $\mathrm{msup}\big((f_1, W_{[0,1]}),(f_2, W_{[0,1]}),(f_3, W_{(1,2]}),(f_4, W_{(1,2]})\big)$.

Example 1 continued



$$\mathrm{msup}\big((f_1, W_{[0,1]}), (f_2, W_{[0,1]}), (f_3, W_{(1,2]}), (f_4, W_{(1,2]})\big)$$

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We can infer from this example that $(\mathbb{R}^{[0,2]}, \{W_{[0,1]}, W_{(1,2]}\})$ *is a Dedekind complete multi-lattice.*

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We also see that the particular multi-supremum in this example is unique.

Remark

$\underset{i\in I}{\operatorname{msup}}(x_i, W_i)$ *is a singleton set if and only if* $\bigcap_{i\in I} W_i$ *is a cone.*

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Besides losing uniqueness of suprema in the venturing from vector lattices to multi-lattices, we also lose the inductive property that vector lattices are closed under finite suprema. Indeed, there exist multi-wedged spaces that are n-multi-lattices but not $(n+1)$-multi-lattices.

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Definition

Let $(E, \mathcal{W})$ and $(F, \mathcal{V})$ be multi-wedged spaces.

Multi-wedged spaces of operators

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$$\mathcal{L}_{\mathcal{W},\mathcal{V}}(E, F) = \{\mathcal{L}_{W,V}(E, F)\colon W \in \mathcal{W}, V \in \mathcal{V}\}.$$

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Proposition

$\big(\mathcal{L}(E, F), \mathcal{L}_{\mathcal{W},\mathcal{V}}(E, F)\big)$ is a multi-wedged space.

Riesz decomposition property

Remark

Since we wish to obtain Riesz-Kantorovich formulas for multi-wedged spaces of operators, we need a natural generalization of the Riesz decomposition property for the multi-wedged setting.

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there exist $z_{ij} \in W_j$ for which

$$x_i = \sum_{j=1}^{n} z_{ij} \quad \text{and} \quad y_j = \sum_{i=1}^{m} z_{ij}.$$

Losing more properties from the classical theory

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There exist (Dedekind complete) multi-lattices that do not even have the $(2,2)$-RDP.

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There exist multi-wedged spaces that have the (m,n)-RDP but not the $(m,n+1)$-RDP.

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Let $(E, \mathcal{W})$ be a multi-wedged space and (F, V) be an ordered vector space that is a Dedekind complete multi-lattice. Consider the multi-wedged space $\left(\mathcal{L}(E,F), \mathcal{L}_{\mathcal{W},\{V\}}(E,F)\right)$. Also consider a multi-bounded above collection $\left(T_i, \mathcal{L}_{W_i,V}(E,F)\right)_{i\in I}$.

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Main theorem continued

Theorem (continued)

If either

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1. $|I| \leq n$ *and* $(E, \mathcal{W})$ *has the* $(2, n)$*-RDP, or*
2. *the cardinality of* I *is arbitrary and* $(E, \mathcal{W})$ *has the* $(2, n)$*-RDP for every* $n \in \mathbb{N}$

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then for $x \in \sum_{i \in I} W_i$,

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then for $x \in \sum_{i \in I} W_i$,

$$\operatorname*{msup}_{i \in I}\big(T_i, \mathcal{L}_{W_i, V}(E, F)\big)(x) =$$

$$\sup\left\{\sum_{i \in I} T_i(y_i) \colon (y_i)_{i \in I} \in \bigoplus_{i \in I} W_i,\ \sum_{i \in I} y_i = x\right\}.$$

Main theorem continued

Theorem (continued)

In particular, under the assumptions of (1) we have that $(\mathcal{L}(E,F), \mathcal{L}_{\mathcal{W},\{V\}}(E,F))$ *is an n-multi-lattice, whereas* $(\mathcal{L}(E,F), \mathcal{L}_{\mathcal{W},\{V\}}(E,F))$ *is a Dedekind complete multi-lattice under the assumptions of (2).*

(1) $|I| \leq n$ and $(E, \mathcal{W})$ has the $(2, n)$-RDP,

(2) the cardinality of I is arbitrary and $(E, \mathcal{W})$ has the $(2, n)$-RDP for every $n \in \mathbb{N}$

A more general case

Remark

This theorem is also valid even if $E \neq \sum_{i \in I} W_i - \sum_{i \in I} W_i$ *and when* V *is a wedge that is not a cone, but the Riesz-Kantorovich formulas get a bit unwieldy.*

Acknowledgement

Acknowledgment

This research was partially funded by the Claude Leon Foundation and by the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS).

The End

Thank you for listening!