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(Pre-)Duals of the space of integral operators

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- The (pre-)dual of the band $(E' \otimes F')^{dd}$
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Definitions and notations

(X, Σ, μ) is a complete σ-finite measure space.
Let E and F be Banach function spaces on (X, Σ, μ). By E^* we denote the Banach space dual of E and by E' the associate space of E (also known as the Köthe dual or order continuous dual E'). It is well-known that the band $(E' \otimes F)^{dd}$, generated by the order continuous finite rank operators in $\mathcal{L}_r(E, F)$, is equal to the collection of all regular (non-singular) integral operators from E into F. We will frequently identify the operators with their kernels and consider $(E' \otimes F)^{dd}$ as a Banach function space. We will write $T_1 \cdot T_2$ to denote the pointwise product of the kernels of T_1 and T_2.
Main Problem of the talk: Describe the kernels of elements of $(E' \otimes F)^{dd}$ and also the elements of its associate space $((E' \otimes F)^{dd})'$.

Integral operators on L^p

It is clear that $(L^{p'} \otimes L^p)^{dd}$ with the regular norm is a Banach function space on $X \times X$ with the Fatou property. For a measurable function f on $X \times X$ we define for $1 \leq p < \infty$ the norm $\|F\|_{\infty,p}$ as follows

$$\|f\|_{\infty,p} = \left\| \left(\int |f(x,y)|^p \, dy \right)^{\frac{1}{p}} \right\|_{\infty}.$$

We denote

$$L_{\infty,p} = \{f \in L_0(X \times X) : \|f\|_{\infty,p} < \infty\}.$$

Given f on $X \times X$ we define the transpose of f by $f^t(x,y) = f(y,x)$. Then $L^t_{\infty,p}$ will denote the collection of all f such that $f^t \in L_{\infty,p}$ and the norm on $L^t_{\infty,p}$ will be defined by $\|f^t\|_{\infty,p}$.

Theorem

Let $1 < p < \infty$. Then $L_{\infty,p'} \cdot L^t_{\infty,p}$ is a product Banach function space isometrically equal to $(L^{p'} \otimes L^p)^{dd}$ and for any $T \in (L^{p'} \otimes L^p)^{dd}$ we have a factorization $T = T_1 \cdot T_2$ with $\|T\|_r = \|T_1\|_{\infty,p'} \|T_2^t\|_{\infty,p}$.

Sketch of Proof

The inclusion $L_{\infty,p'} \cdot L^t_{\infty,p} \subset (L^{p'} \otimes L^p)^{dd}$ is a consequence of Hölder's inequality. Now let $T \in (L^{p'} \otimes L^p)^{dd}$. Then we can assume that $0 \leq T$ and $\|T\| = 1$. Let $\epsilon > 0$. Then by Gagliardo's converse of the Schur test for positive linear operators there exists $0 < f_0 \in L_p$ with $\|f_0\|_p = 1$ such that $T^*(Tf_0)^{p-1} \leq (1+\epsilon)f_0^{p-1}$. Define now $T_1(x,y) = T(x,y)^{\frac{1}{p'}} f_0(y)^{\frac{1}{p'}} (Tf_0(x))^{-\frac{1}{p'}}$ and $T_2(x,y) = T(x,y)^{\frac{1}{p}} f_0(y)^{-\frac{1}{p'}} (Tf_0(x))^{\frac{1}{p'}}$. Then clearly $T(x,y) = T_1(x,y)T_2(x,y)$. Moreover

$$\int T_1(x,y)^{p'} \, d\mu(y) = Tf_0(x)(Tf_0(x))^{-1} = 1 \text{ a.e.}$$

and

$$\int T_2(x,y)^p \, d\mu(x) = T^*(Tf_0)^{p-1}(y) \cdot f_0(y)^{1-p} \leq 1 + \epsilon \text{ a.e.}$$

This shows that $T_1 \in L_{\infty,p'}$ and $T_2^t \in L^p_{\infty,p}$ and $\|T_1\|_{\infty,p'}\|T_2^t\|_{\infty,p} \leq 1+\epsilon$. Hence $\|T\|_r = \inf\{||T_1\|_{\infty,p'}\|T_2^t\|_{\infty,p} : |T(x,y)| = |T_1(x,y)T_2(x,y)|, T_1 \in L_{\infty,p'}, T_2^t \in L^t_{\infty,p}\}$. This shows that $L_{\infty,p'} \cdot L^t_{\infty,p}$ is a product Banach function space isometrically equal to $(L^{p'} \otimes L^p)^{dd}$. That the infimum is a minimum follows from one of our theorems on Cesaro convergence and Komlos' Theorem.

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Remark: One can rewrite the above factorization as $(L^{p'} \otimes L^p)^{dd} = (L_{\infty,1})^{\frac{1}{p'}}(L^t_{\infty,1})^{\frac{1}{p}}$. In this form the above theorem is a version of Pisier's result (1994) that $\mathcal{L}_r(\ell_p(n)) = \mathcal{L}(\ell_\infty(n))^{\frac{1}{p'}}\mathcal{L}(\ell_1(n))^{\frac{1}{p}}$. Pisier 's result was already partially anticipated by Akcoglu, Baxter and Lee (1991).

The Associate space $((L^{p'} \otimes L^p)^{dd})'$

Form Lozanovskii's duality theorem we get

$$((L^{p'} \otimes L^p)^{dd})' = (L'_{\infty,1})^{\frac{1}{p'}}((L^t_{\infty,1})')^{\frac{1}{p}} = (L_{1,\infty})^{\frac{1}{p'}}(L^t_{1,\infty})^{\frac{1}{p}} = L_{p',\infty}L^t_{p,\infty}.$$

Let $\mathcal{K}_r(L^p)$ denote the closure of $L^{p'} \otimes L^p$ in $(L^{p'} \otimes L^p)^{dd}$. Then $\mathcal{K}_r(L^p)$ consists exactly of the elements of order continuous norm in $(L^{p'} \otimes L^p)^{dd}$, so also $(\mathcal{K}_r(L^p))^* = L_{p',\infty}L^t_{p,\infty}$.

Connection to the positive projective tensor product

Recall that the positive projective tensor product $L^{p'}\hat{\otimes}_{|\pi|}L^p$ is the completion of the Riesz subspace generated by $L^{p'} \otimes L^p$ with respect to the norm

$$\|f\|_{|\pi|} = \inf\{\|g\|_{p'}\|h\|_p : |f| \leq g \otimes h\}.$$

Let $\mathcal{F}_r(L^p)$ denote the ideal generated by $L^{p'} \otimes L^p$ with the norm $\|\cdot\|_{|\pi|}$.

Theorem

$\mathcal{F}_r(L^p)$ is a Banach function space with the Fatou property.

Corollary

$\mathcal{F}_r(L^p) = ((L^{p'} \otimes L^p)^{dd})' = \mathcal{K}_r(L^p)^* = L_{p',\infty}L^t_{p,\infty}.$

The positive (Fremlin) projective tensor product

Let now E and F be Banach function spaces. Then the positive projective tensor product $E\hat{\otimes}_{|\pi|}F$ is the completion of the Riesz subspace generated by $E \otimes F$ with respect to the norm

$$\|f\|_{|\pi|} = \inf\{\sum_{k=1}^{n} \|g_k\|_E \|h_k\|_F : |f| \leq \sum_{k=1}^{n} g_k \otimes h_k\}.$$

Fremlin proved for $X = [0,1]$ with Lebesgue measure:

1. $L^1\hat{\otimes}_{|\pi|}L^1 = L^1([0,1]^2)$.
2. $L^2\hat{\otimes}_{|\pi|}L^2$ has a Fatou norm (i.e. a l.s.c. norm), but is not Dedekind complete.
3. $C(K_1)\hat{\otimes}_{|\pi|}C(K_2) = C(K_1 \times K_2)$.

Order Completeness

Here we collect some additional old and new facts about the Dedekind completeness of $E\hat{\otimes}_{|\pi|}F$.

1. $\ell^\infty\hat{\otimes}_{|\pi|}\ell^\infty$ is not Dedekind complete. Reason: $\ell^\infty = C(\beta\mathbb{N})$, so $\ell^\infty\hat{\otimes}_{|\pi|}\ell^\infty = C(\beta\mathbb{N} \times \beta\mathbb{N})$, by item 3 of the previous slide. Now it is well-known that $\beta\mathbb{N} \times \beta\mathbb{N}$ is not Stonian (look at the closure of $\mathbb{N} \times \mathbb{N}$ in $\beta\mathbb{N} \times \beta\mathbb{N}$). In fact:
2. If K_1 and K_2 Stonian compact Hausdorff spaces, then $K_1 \times K_2$ is Stonian $\iff$ K_1 or K_2 is finite, i.e., $C(K_1)\hat{\otimes}_{|\pi|}C(K_2)$ is Dedekind complete implies that K_1 or K_2 is finite.
3. For $X = [0,1]$ with Lebesgue measure, $L^p\hat{\otimes}_{|\pi|}L^q$, $1 < q \leq p < \infty$, is not Dedekind complete.

The space $E\hat{\bigotimes}_{|\pi|}F$

Recall that$E$ is called p–convex for $1 \leq p \leq \infty$ if there exists a constant M such that for all $f_1, \ldots, f_n \in E$

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_E \leq M \left(\sum_{k=1}^{n} \|f_k\|_E^p \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty$$

or $\| \sup |f_k| \|_E \leq M \cdot \max_{1 \leq k \leq n} \|f_k\|_E$ if $p = \infty$. Similarly E is called p–concave for $1 \leq p \leq \infty$ if there exists a constant M such that for all $f_1, \ldots, f_n \in E$

$$\left(\sum_{k=1}^{n} \|f_k\|_E^p \right)^{\frac{1}{p}} \leq M \left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_E \text{ if } 1 \leq p < \infty$$

and $\max_{1 \leq k \leq n} \|f_k\|_E \leq M \cdot \| \sup |f_k| \|_E$ if $p = \infty$.

Fatou properties

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ *and with convexity constants equal to 1, then for all* $f \in E\hat{\otimes}_{|\pi|}F$

$$\|f\|_{|\pi|} = \inf\{\|g\|_E \|h\|_F : |f| \leq g \otimes h\}.$$

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ *and with with convexity constants equal to 1, then* $E\hat{\otimes}_{|\pi|}F$ *has a Fatou norm (i.e., the norm is lower semi-continuous).*

In fact we have slightly more:

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1, then $0 \leq f_n(x,y) \uparrow$ in $E \hat{\otimes}_{|\pi|} F$ with $\sup \|f_n\|_{|\pi|} = 1$ implies that there exists $0 \leq g \in E$ and $0 \leq h \in F$ such that $f_n \leq g \otimes h$ for all n and $\|g \otimes h\|_{|\pi|} = 1$.

In fact we have slightly more:

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Assume E is p-convex and F is q-convex with $\frac{1}{p}+\frac{1}{q}\leq 1$ and with convexity constants equal to 1, then $0\leq f_n(x,y)\uparrow$ in $E\hat{\otimes}_{|\pi|}F$ with $\sup\|f_n\|_{|\pi|}=1$ implies that there exists $0\leq g\in E$ and $0\leq h\in F$ such that $f_n\leq g\otimes h$ for all n and $\|g\otimes h\|_{|\pi|}=1$.

As in the L^p case we now consider the ideal $\mathcal{F}_r(E,F)$ generated by $E\hat{\otimes}_{|\pi|}F$ with the norm

$$\|f\|_{|\pi|}=\inf\{\|g\|_E\|h\|_F : |f|\leq g\otimes h\}.$$

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1, then the ideal $\mathcal{F}_r(E, F)$ generated by $E\hat{\otimes}_{|\pi|}F$ with the above norm has the Fatou property. In particular $\mathcal{F}_r(E, F)$ is a Banach function space.

We now recall some general duality facts. The dual space of $E\hat{\otimes}_{|\pi|}F$ is the Banach lattice $\mathcal{L}_r(E, F^*)$ of all regular operators from E into F^*. The conditions that E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$, imply that F^* is q' concave and that $q' \leq p$.

An easy consequence of the previous result is:

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1, then the associate space $\mathcal{F}_r(E,F)'$ of the Banach function space $\mathcal{F}_r(E,F)$ is isometric with the band of $(E' \otimes F')^{dd}$ of regular integral operators from E into F'.

To get a more explicit description of the kernels of the regular integral operators from E into F' we identify $\mathcal{F}_r(E,F)$ as a Calderon-Lozanovskii space. For a measurable function f on $X \times X$ we define the norm $\|f\|_{E,\infty}$ as follows

$$\|f\|_{E,\infty} = \| \|f_x(\cdot)\|_\infty \|_E .$$

We denote

$$L_{E,\infty} = \{f \in L_0(X \times X) : \|f\|_{E,\infty} < \infty\}.$$

Given f on $X \times X$ we define as before the transpose of f by $f^t(x,y) = f(y,x)$. Then $L^t_{F,\infty}$ will denote the collection of all f such that $f^t \in L_{F,\infty}$ and the norm on $L^t_{F,\infty}$ will be defined by $\|f^t\|_{F,\infty}$. With this notations we have

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1, then $\mathcal{F}_r(E,F) = L_{E,\infty} \cdot L^t_{F,\infty}$ (as a pointwise product Banach function space).

Corollary

Let E and F be as in the above theorem. Then the band $(E' \otimes F')^{dd}$ of regular integral operators from E into F′ is equal to $(L_{E,\infty} \cdot L^t_{F,\infty})'$.

What is $(L_{E,\infty} \cdot L^t_{F,\infty})'$?

Our first description is a corollary of a proposition of my paper on product BFS.

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1. Then
$(E' \otimes F')^{dd} = M(L_{E,\infty}, (L^t_{F,\infty})') = M(L^t_{F,\infty}, (L_{E,\infty})')$.

To find the associate spaces of those mixed norm spaces we recall a result (unpublished?) of Wim Luxemburg.

Theorem

Let ρ_1, ρ_2 be Banach function norms and assume ρ_2 has the Fatou property. Then

$$(\rho_1 \circ \rho_2)' = \rho_1' \circ \rho_2'.$$

From Luxemburg's result we see that $(L^t_{F,\infty})' = L^t_{F',1}$ and $L_{E,\infty})' = L_{E',1}$. Therefore we have

Theorem

Assume E is p-convex and F is q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1. Then $(E' \otimes F')^{dd} = M(L_{E,\infty}, L^t_{F',1}) = M(L^t_{F,\infty}, L_{E',1})$.

In practice this description is not that useful and doesn't recover the $E = F = L^p$ case, mentioned earlier. An inspection of that earlier case shows that the p-concavication was used. Recall if E is p-convex with $p > 1$ and convexity constant equal to 1, then the p-concavication E^p is the Banach function space consisting of measurable functions f with $|f|^p \in E$ and norm $\|f\|_{E^p} = \||f|^p\|_E^{\frac{1}{p}}$.

The Calderon-Lozanovski space description

Theorem

Let E be p-convex and F be q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$ and with convexity constants equal to 1. Then
$\mathcal{F}_r(E,F) = (L_{E^p,\infty})^{\frac{1}{p}} \cdot (L^t_{F^{p'},\infty})^{\frac{1}{p'}} = (L_{E^{q'},\infty})^{\frac{1}{q'}} \cdot (L^t_{F^q,\infty})^{\frac{1}{q}}$

Corollary

Let E and F be as in the above theorem. Then the band $(E' \otimes F')^{dd}$ of regular integral operators from E into F' is equal to

$$(L_{(E^p)',1})^{\frac{1}{p}} \cdot (L^t_{(F^{p'})',1})^{\frac{1}{p'}} = (L_{(E^{q'})',1})^{\frac{1}{q'}} \cdot (L^t_{(F^q)',1})^{\frac{1}{q}}.$$

Extrapolation and Interpolation

The last theorem can be viewed as an extrapolation result. Let $0 \leq T : E \to F'$ be an integral operator, where E and F are as above. Then $T = T_1^{\frac{1}{p}} \cdot T_2^{\frac{1}{p'}}$, where $T_1 : L^\infty \to (E^p)'$ and $T_2 : F^{p'} \to L^1$ (and a similar extrapolation involving the q's). We have a converse of the previous theorem, by tracing back through the proof.

Theorem

Let E be p-convex and F be q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$. Assume $T_1 : L^\infty \to (E^p)'$ and $T_2 : F^{p'} \to L^1$ are regular integral operators, then $T = T_1^{\frac{1}{p}} \cdot T_2^{\frac{1}{p'}} : E \to F$ is a regular integral operator.

Extrapolation and Interpolation continued

The special case $T = T_1 = T_2$ is an interpolation result.

Corollary

Let E be p-convex and F be q-convex with $\frac{1}{p} + \frac{1}{q} \leq 1$. Assume $T : L^\infty \to (E^p)'$ and $T : F^{p'} \to L^1$ are regular integral operators, then $T : E \to F$ is a regular integral operator.

Open problem: Can we extend the above extrapolation and interpolation results to arbitrary regular operators?

The special case $E = L^p$ and $F = L^q$

In this case the condition $\frac{1}{p} + \frac{1}{q} < 1$ implies that $1 \leq q' < p$, so that the main result describes the band $(L^{p'} \otimes L^{q'})^{dd}$ of regular integral operators from L^p into $L^{q'}$, where $1 \leq q' < p$. In this case $E^p = L^1$ and $F^{p'} = L^{\frac{q}{p'}}$ so that $T = T_1^{\frac{1}{p}} \cdot T_2^{\frac{1}{p'}}$, where $T_1 \in L_{\infty,1}$ and $T_2^t \in L_{\infty,\frac{q}{p'}}$, or $T_1 : L^\infty \to L^\infty$ and $T_2 : L^{\frac{q}{p'}} \to L^1$.

Factorization of $\mathcal{L}_r(L^p, L^{q'})$, where $q' < p$

Recall first that the space $M(L^p, L^{q'})$ of bounded multiplication operators from L^p into $L^{q'}$ can be identified with L^r, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Maurey proved that

$$\mathcal{L}_r(L^p, L^{q'}) = M(L^p, L^{q'}) \circ \mathcal{L}_r(L^p, L^p).$$

Hence also

$$(L^{p'} \otimes L^{q'})^{dd} = M(L^p, L^{q'}) \circ (L^{p'} \otimes L^p)^{dd}.$$

At the level of the kernels this shows that

$$(L^{p'} \otimes L^{q'})^{dd} = L_{r,\infty} \cdot (L^{p'} \otimes L^p)^{dd},$$

or $M((L^{p'} \otimes L^{q'})^{dd}, (L^{p'} \otimes L^p)^{dd}) = L_{r,\infty}$.

Factorization of integral operators

We discuss now the problem of factorization of integral operators from E into F'. It is easy to see that we can factor rank one operators (and thus finite rank operators) from E into F' through E, followed by a multiplication operator, if and only if $E \cdot M(E, F') = F'$. As it happens, this is true under almost the exact same hypotheses as before.

Theorem

Let E and F be Banach function spaces such that there exists $1 < p < \infty$ such that E is p-convex and F is p'-convex with convexity constants equal to 1 and assume E has the Fatou property. Then $E \cdot M(E, F')$ is a product Banach function space and $E \cdot M(E, F') = F'$.

Factorization of integral operators

The result on the previous slide leads to the inclusions:

$$E' \otimes F' \subset M(E,F') \circ (E' \otimes E)^{dd} \subset (E' \otimes F')^{dd}.$$

We can again replace $M(E,F') \circ (E' \otimes E)^{dd}$ with $L_{M(E,F'),\infty} \cdot (E' \otimes E)^{dd}$ and now we have the following open problems:

- Is
$$L_{M(E,F'),\infty} \cdot (E' \otimes E)^{dd} = (E' \otimes F')^{dd}?$$
or
$$M(E,F') \circ (E' \otimes E)^{dd} = (E' \otimes F')^{dd}?$$
- Is the “factorization norm” on $M(E,F') \circ (E' \otimes E)^{dd}$ equal to the regular operator norm?
- Is $\mathcal{L}_r(E,F') = M(E,F')\mathcal{L}_r(E,E)$ under the same hypotheses?

Some more open problems

Besides the open problems about factorization we have:

1. Can one describe the integral operators from L^p to L^q, where $q > p$? Note in this case the operator norm is not order continuous. The case $q = \infty$ is easy. In that case every bounded operator is an integral operator.
2. Is it true that if E and F are p-convex for some $p > 1$ and $E\hat{\otimes}_{|\pi|}F$ is Dedekind complete, then E or F is atomic.