Fatou's Lemma, Galerkin Approximations and the Existence of Walrasian Equilibria in Infinite Dimensions

M. Ali Khan[1]   Nobusumi Sagara[2]

[1]Department of Economics, Johns Hopkins University

[2]Department of Economics, Hosei University

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Theorem (Schmeidler 1970)

Let $\{f_n\}$ be a sequence of integrable functions on a measure space T with values in $\mathbb{R}^k_+$ for which $\int f_n d\mu \to x \in \mathbb{R}^k_+$. Then there exists an integrable function $f : T \to \mathbb{R}^k_+$ such that:

(i) $f(t)$ *is a limit point of* $\{f_n(t)\}$ *a.e.* $t \in T$.

(ii) $\int f d\mu \leq x$.

- When $k = 1$, the result is a form of Fatou's lemma.
  - It cannot be proved by a successive application of Fatou's lemma k times.
  - For the proof, Lyapunov's convexity theorem is effectively used for the nonatomic parts of the measure space.

1. A failure of Lyapunov's convexity theorem in infinite dimensions $\to$ the need to strengthen the notion of nonatomicity.
2. In infinite-dimensional vector spaces without order structures, the inequality must be changed into an inclusion form.

Outline of the Talk

1. Fatou's lemma in Banach spaces without order structures
   - ▷ Sequence of Bochner integrable functions
   - ▷ Introduction of saturated measure spaces
   - ▷ Necessity and sufficiency of saturation for the Fatou property
2. Application to large economies: the existence of Walrasian equilibria with an infinite-dimensional commodity spaces
   - ▷ the existence result in finite dimensions
   - ▷ Galerkin approximation of infinite-dimensional commodity spaces
   - ▷ Fatou's lemma in infinite dimensions

Saturated Measure Spaces (Keisler & Sun 2009)

- A measure space (T, Σ, μ) is *essentially countably generated* $\overset{\text{def}}{\Longleftrightarrow}$ Σ is generated by a countable number of subsets of T together with the null sets.
  - (T, Σ, μ) is *essentially uncountably generated* $\overset{\text{def}}{\Longleftrightarrow}$ Σ it is not essentially countably generated.
  - $\Sigma_S = \{A \cap S \mid A \in \Sigma\}$: the σ-algebra restricted to $S \in \Sigma$.
  - $L^1_S(\mu)$: the space of μ-integrable functions on (S, Σ_S) whose element is a restriction of a function in $L^1(\mu)$ to S.
- A finite measure space (T, Σ, μ) is *saturated* $\overset{\text{def}}{\Longleftrightarrow}$ $L^1_S(\mu)$ is nonseparable $\forall S \in \Sigma$ with $\mu(S) > 0$ $\Longleftrightarrow$ Σ_S is essentially uncountably generated $\forall S \in \Sigma$ with $\mu(S) > 0$.
  - Saturation involves a “rich” σ-algebra with an uncountable number of measurable sets, which is a strengthened notion of nonatomicity.
    - For every uncountable cardinal κ, the product spaces $\{0,1\}^\kappa$ with the uniform measure and $[0,1]^\kappa$ with the Lebesgue measure are saturated probability spaces.
    - $\{0,1\}^{\mathbb{N}}$, $[0,1]^{\mathbb{N}}$: nonatomic probability spaces, but not saturated.

Saturation and Lyapunov's Convexity Theorem

Theorem (Khan & Sagara 2013)

Let E be a separable Banach space. If (T, Σ, μ) is saturated, then for every μ-continuous vector measure $m : \Sigma \to E$, then its range $m(\Sigma)$ is w-compact and convex. Conversely, if every μ-continuous vector measure $m : \Sigma \to E$ has the w-compact convex range, then (T, Σ, μ) is saturated whenever E is infinite dimensional.

- Saturation is not only sufficient, but also necessary for Lyapunov's convexity theorem holds in separable Banach spaces.
- a complete characterization of saturation in terms of Lyapunov's convexity theorem.
- The result is further extended to sequentially complete, separable, locally convex spaces by Khan & Sagara (2015, 2016), Sagara (2017).
  - ▷ Dual spaces with w^*-topology of a separable Banach space are covered.

Notations

- $L^1(\mu, E)$: the space of E-valued Bochner integrable functions on a finite measure space (T, Σ, μ).
  - ▷ A sequence $\{f_n\}$ in $L^1(\mu, E)$ is *well-dominated* $\overset{\text{def}}{\Longleftrightarrow}$ there is an integrably bounded, w-compact-valued multifunction $K : T \twoheadrightarrow E$ such that $f_n(t) \in K(t)$ a.e. $t \in T$ $\forall n$.
- the *weak upper limit* of a sequence $\{x_n\}$ in E:

$$w\text{-Ls}\{x_n\} = \left\{ x \in E \;\middle|\; \begin{array}{l} \exists \text{ a subsequence } \{x_{n_i}\} \subset \{x_n\} : \\ x = w\text{-}\lim\limits_{i \to \infty} x_{n_i} \end{array} \right\}.$$

- $\Gamma : T \twoheadrightarrow E$: a multifunction (with nonempty values).
  - ▷ $f : T \to E$ is a *selector* of Γ $\overset{\text{def}}{\Longleftrightarrow}$ $f(t) \in \Gamma(t)$ a.e. $t \in T$.
  - ▷ $\mathcal{S}^1_\Gamma$: the set of Bochner integrable selectors of Γ.
    - ◇ $\mathcal{S}^1_\Gamma \neq \emptyset$ if E is separable, and Γ is measurable and integrably bounded.
- The *Bochner integral* of Γ:

$$\int \Gamma d\mu := \left\{ \int f d\mu \mid f \in \mathcal{S}^1_\Gamma \right\}.$$

Definition

Let $\{f_n\}$ be a sequence in $L^1(\mu, E)$ with w-Ls$\{\int f_n d\mu\} \neq \emptyset$.

(i) $\{f_n\}$ satisfies the *weak upper closure property* if there exists $f \in L^1(\mu, E)$ such that:

(a) $f(t) \in w\text{-Ls}\{f_n(t)\}$ a.e. $t \in T$.

(b) $\displaystyle\int f d\mu \in w\text{-Ls}\left\{\int f_n d\mu\right\}$.

(ii) $\{f_n\}$ satisfies the *Fatou property* if:

$$w\text{-Ls}\left\{\int f_n d\mu\right\} \subset \int w\text{-Ls}\{f_n\} d\mu.$$

Theorem (Khan & Sagara 2014)

Let (T, Σ, μ) *be a nonatomic finite measure space and* E *be an infinite-dimensional Banach space. Then the following conditions are equivalent.*

(i) (T, Σ, μ) *has the saturation property.*

(ii) *Every well-dominated sequence in* $L^1(\mu, E)$ *has the weak upper closure property.*

(iii) *Every well-dominated sequence in* $L^1(\mu, E)$ *has the Fatou property.*

- For an exact Fatou's lemma to be true in Banach spaces, saturation is not only sufficient, but also necessary.
- The result can be formulated in dual spaces of a separable Banach spaces with Gelfand integrals.

Khan, M. A., N. Sagara and T. Suzuki. An exact Fatou lemma for Gelfand integrals: A characterization of the Fatou property, *Positivity* **20** (2016), 343–354.

- large economies: a prototype of perfect competition with the continuum of agents modeled as a nonatomic finite measure space.
  - ▷ Aumann (1966): the existence of Walrasian equilibria in large economies without any convexity assumption on preferences in the setting of finite-dimensional commodity spaces.
- a failure of the Lyapunov convexity theorem in infinite dimensions.
  - ▷ nonconvexity of the aggregate demand multifunction $\Rightarrow$ inapplicability of fixed point theorems.
  - ▷ the inevitability of the convexity assumptions on preferences.
- a need to strengthen the notion of nonatomicity so that the Lyapunov convexity theorem holds in infinite-dimensional commodity spaces.
  - ▷ Rustichini & Yannelis (1991): the assumption “many more agents than commodities”.
  - ▷ Podczeck (1997): a condition on the nonatomic disintegration of the measure space of agents.
  - ▷ Khan & Sagara (2016): relaxation and purification under saturation.

Finite-Dimensional Truncation: Bewley (1972)

- Finite agent economies with commodity space L^∞ under convex preferences.
  - ▷ the first application of Galerkin approximations to GE.
- the idea of the procedure:
  1. Take a net of finite dimensional vector subspaces of L^∞ directed by set inclusion and consider a net of truncated subeconomies in which a finite-dimensional vector subspace is a commodity space.
  2. Each truncated subeconomy has equilibria by the classical finite-dimensional result of Arrow–Debreu.
  3. Take the limit of the net of equilibria. Then the limit corresponds to a Walrasian equilibria in the original L^∞ economy.
- a natural approach to the existence of Walrasian equilibria also in large economies with a separable Banach space and L^∞ without convex preferences.
  - ▷ the combination of the Galerkin approximation with finite-dimensional projections and Fatou's lemma in infinite dimensions.
  - ▷ an alternative technique to the existence result which employs fixed point theorems in infinite dimensions (cf. Khan & Yannelis 1991).

- A *Galerkin approximation scheme* of a Banach space E is a sequence $\{V^n\}_{n\in\mathbb{N}}$ of finite-dimensional subspaces of E such that for every $x \in E$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in V^n$ for each $n \in \mathbb{N}$ and $x_n \to x$.

Theorem

Let E be a separable Banach space and $\{V^n\}_{n\in\mathbb{N}}$ be a Galerkin approximation scheme of E such that $V^1 \subset V^2 \subset \cdots$ and $\overline{\bigcup_{n\in\mathbb{N}} V^n}^{\|\cdot\|} = E$. If P_n is a continuous projection of E onto V^n, then for every $x \in E$ the sequence $\{P_n x\}_{n\in\mathbb{N}}$ contains a subsequence converging weakly to x.

Large Economies in a Banach Space

- (T, Σ, μ): the set of agents.
  - ▷ a finite measure space.
- E: a commodity space.
  - ▷ an ordered Banach space.
- $X(t) \subset E$: a consumption set for agent $t \in T$.
- $\succsim(t) \subset X(t) \times X(t)$: a preference relation on $X(t)$ for agent t.
  - ▷ a complete, transitive binary relation on $X(t)$.
- $\omega(t) \in X(t)$: an initial endowment of agent t.
  - ▷ $\omega \in L^1(\mu, E)$.
- $\mathcal{E} = \{(T, \Sigma, \mu), X, \succsim, \omega\}$: an economy.
- E^*: a price space.
- A function $f \in L^1(\mu, E)$ is an *allocation* with free disposal for $\mathcal{E} \overset{\text{def}}{\Longleftrightarrow}$ $f(t) \in X(t)$ a.e. $t \in T$ and $\int f d\mu \le \int \omega d\mu$.
- A price-allocation pair (p, f) is a *Walrasian equilibrium* with free disposal for $\mathcal{E} \overset{\text{def}}{\Longleftrightarrow}$ for a.e. $t \in T$: $\langle p, f(t)\rangle \le \langle p, \omega(t)\rangle$ and $\langle p, x\rangle > \langle p, \omega(t)\rangle$ whenever $x \succ(t) f(t)$.

Assumption

(i) $X : T \twoheadrightarrow E_+$ is an integrably bounded multifunction with weakly compact, convex values.

(ii) $\operatorname{gph} X \in \Sigma \otimes \mathrm{Borel}(E, w)$.

(iii) $\forall t \in T\ \exists z(t) \in X(t)$: $\omega(t) - z(t) \in \operatorname{int} E_+$.

(iv) $\succsim(t)$ is weakly closed in $X(t) \times X(t)\ \forall t \in T$.

(v) $\{(t, x, y) \in T \times E \times E \mid x \succsim(t) y\} \in \Sigma \otimes \mathrm{Borel}(E, w) \otimes \mathrm{Borel}(E, w)$.

(vi) If $x \in X(t)$ is a satiation point for $\succsim(t)$, then $x \geq \omega(t)$; if $x \in X(t)$ is not a satiation point for $\succsim(t)$, then $x \in \overline{\{y \in X(t) \mid y \succ(t) x\}}^{w}$.

- Continuous preferences on the weakly compact consumption set have a satiation point.
- No monotonicity assumption on preferences.
- $\operatorname{int} E_+ \neq \emptyset$ is an inevitable assumption to deal with infinite dimensionality in general equilibrium theory.

Existence of Walrasian Equilibria with Free Disposal

Theorem

Let (T, Σ, μ) be a saturated finite measure space and E be an ordered separable Banach space such that $\operatorname{int} E_+ \neq \emptyset$. Then for every economy $\mathcal{E}$ satisfying Assumption, there exists a Walrasian equilibrium (p, f) with free disposal for $\mathcal{E}$ with $p \in E_+^* \setminus \{0\}$.

An Existence Theorem in Finite Dimensions

Auxiliary Theorem (Khan & Yannelis 1991)

Let (T, Σ, μ) be a nonatomic finite measure space. Suppose that the economy $\mathcal{E}_k$ with a finite-dimensional commodity space $\mathbb{R}^k$ satisfies the following conditions.

(i) $X : T \twoheadrightarrow \mathbb{R}^k_+$ *is an integrably bounded multifunction with compact, convex values.*

(ii) $\mathrm{gph}\, X \in \Sigma \otimes \mathrm{Borel}(\mathbb{R}^k)$.

(iii) $\forall t \in T\ \exists z(t) \in \mathbb{R}^k_+ : \omega(t) - z(t) \in \mathbb{R}^k_{++}$.

(iv) $\precsim(t)$ *is closed in* $\mathbb{R}^k_+ \times \mathbb{R}^k_+$ $\forall t \in T$.

(v) $\{(t, x, y) \in T \times \mathbb{R}^k \times \mathbb{R}^k \mid x \precsim(t)\, y\} \in \Sigma \otimes \mathrm{Borel}(\mathbb{R}^k) \otimes \mathrm{Borel}(\mathbb{R}^k)$.

Then there exists a Walrasian equilibrium (p, f) with free disposal for $\mathcal{E}_k$ with $p \in \mathbb{R}^k_+ \setminus \{0\}$.

- This is not covered by Aumann (1966) with monotone preferences.

Sketch of the Proof: (Step 1)

- $\{V^n\}_{n\in\mathbb{N}}$: a Galerkin approximation scheme of E such that $V^1 \subset V^2 \subset \cdots$ with $\overline{\bigcup_{n\in\mathbb{N}} V^n}^{\|\cdot\|} = E$.
- P_n: a continuous projection of E onto V^n.
  - ▷ $V^n_+ = V^n \cap E_+$ is a positive cone of V^n and $P_n : E \to V^n$ is a positive linear operator.
- Construct a sequence of economies with a finite-dimensional truncation as follows:
  - ▷ $X^n(t) := P_n(X(t)) \subset P_n(E_+) \subset V^n_+$: a consumption set of each agent restricted to the finite-dimensional commodity space V^n.
  - ▷ $\succsim_n(t)$: the restriction of the preference $\succsim(t)$ to $X^n(t)$, i.e., $\succsim_n(t) := \succsim(t) \cap (X^n(t) \times X^n(t))$.
  - ▷ $\omega_n(t) = P_n\omega(t) \in X_n(t)$: the initial endowment with $\omega_n \in L^1(\mu, V^n)$.
  - ▷ $\mathcal{E}_n = \{(T, \Sigma, \mu), X_n, \succsim_n, \omega_n\}$: a finite-dimensional truncation of economy $\mathcal{E}$ with commodity space V^n conformed with the Galerkin approximation scheme.

- The finite-dimensional truncated economy $\mathcal{E}_n$ of $\mathcal{E}$ satisfies:
  - (i$_n$) $X^n : T \twoheadrightarrow V^n_+$ is an integrably bounded multifunction with compact, convex values.
  - (ii$_n$) $\operatorname{gph} X^n \in \Sigma \otimes \mathrm{Borel}(V^n)$.
  - (iii$_n$) $\forall t \in T\ \exists z_n(t) \in X^n(t)$: $\omega_n(t) - z_n(t) \in \operatorname{int} V^n_+$.
  - (iv$_n$) $\succsim_n(t)$ is a closed subset of $V^n_+ \times V^n_+\ \forall t \in T$.
  - (v$_n$) $\{(t, x, y) \in T \times V^n \times V^n \mid x \succsim(t)\, y\} \in \Sigma \otimes \mathrm{Borel}(V^n) \otimes \mathrm{Borel}(V^n)$.
- By Auxiliary Theorem, there is a Walrasian equilibrium $(q_n, f_n) \in ((V^n)^*_+ \setminus \{0\}) \times L^1(\mu, V^n)$ with free disposal for $\mathcal{E}_n$ $\forall n \in \mathbb{N}$.

Sketch of the Proof: (Step 2)

- By the Fatou's lemma, there exist $f, g \in L^1(\mu, E)$ such that:

$$f(t), g(t) \in X(t), \quad (f(t), g(t)) \in w\text{-Ls}\,\{(f_n(t), \omega_n(t))\} \quad \text{a.e. } t \in T,$$
$$\left(\int f d\mu, \int g d\mu\right) \in w\text{-Ls}\left\{\left(\int f_n d\mu, \int \omega_n d\mu\right)\right\}.$$

  - ▷ Along a subsequence: $(\int f_n d\mu, \int \omega_n d\mu) \to (\int f d\mu, \int g d\mu)$ weakly and for a.e. $t \in T$: $\omega_n(t) \to g(t)$ weakly.
  - ▷ For a.e. $t \in T$: $\omega_n(t) = P_n\omega(t) \to \omega(t)$ weakly. $\therefore g(t) = \omega(t)$ a.e. $t \in T$ and $\int \omega_n d\mu \to \int \omega d\mu$ weakly.
  - ▷ Since $\int f_n d\mu \le \int \omega_n d\mu\ \forall n \in \mathbb{N}$, at the limit: $\int f d\mu \le \int \omega d\mu$.
  - ▷ f is an allocation with free disposal for $\mathcal{E}$.

- Since $0 \ne q_n \in (V^n)^* \subset E^*$, by the Krein–Rutman theorem, q_n can be extended as a continuous positive linear functional to E.
  - ▷ Normalize equilibrium price for $\mathcal{E}_n$ such that:

$$p_n = \frac{q_n}{\|q_n\|} \in \Delta^* := \{p \in E^*_+ \mid \|p\| = 1\}.$$

  - ▷ There is a subsequence such that $p_n \to p \in \Delta^*$ weakly*.
  - ▷ $(p, f) \in \Delta^* \times L^1(\mu, E)$ is a Walrasian equilibrium with free disposal for $\mathcal{E}$.

References

[1] Aumann, R. J. Existence of competitive equilibria in markets with a continuum of traders, *Econometrica* **34** (1966), 1–17.

[2] Bewley, T. F. Existence of equilibria in economies with infinitely many commodities, *J. Econom. Theory* **4** (1972), 514–540.

[3] Keisler, H. J. and Y. N. Sun. Why saturated probability spaces are necessary, *Adv. Math.* **221** (2009), 1584–1607.

[4] Khan, M. A. and M. K. Majumdar. Weak sequential convergence in $L_1(\mu, X)$ and an approximate version of Fatou's lemma, *J. Math. Anal. Appl.* **114** (1986), 569–573.

[5] Khan, M. A. and N. Sagara. "Maharam-types and Lyapunov's theorem for vector measures on Banach spaces", *Illinois J. Math.* **57** (2013), 145–169.

[6] Khan, M. A. and N. Sagara. Weak sequential convergence in $L^1(\mu, X)$ and an exact version of Fatou's lemma, *J. Math. Anal. Appl.* **412** (2014), 554–563.

[7] Khan, M. A. and N. Sagara. "Maharam-types and Lyapunov's theorem for vector measures on locally convex spaces with control measures", *J. Convex Anal.* **22** (2015), 647–672.

[8] Khan, M. A. and N. Sagara. "Maharam-types and Lyapunov's theorem for vector measures on locally convex spaces without control measures, *Pure Appl. Func. Anal.* **1** (2016), 47–62.

[9] Khan, M. A. and N. Sagara. Relaxed large economies with infinite-dimensional commodity spaces: The existence of Walrasian equilibria, *J. Math. Econom.* **67** (2016), 95–107.

[10] Khan, M. A. and N. C. Yannelis. Equilibria in markets with a continuum of agents and commodities, in: M. A. Khan and N. C. Yannelis (eds.), *Equilibrium Theory in Infinite Dimensional Spaces*, Springer, Berlin, pp. 233–248, 1991.

[11] Lee, S. Competitive equilibrium with an atomless measure space of agents and infinite dimensional commodity spaces without convex and complete preferences, *Hitotsubashi J. Econom.* **54** (2013), 221–230.

[12] Loeb, P. A. and Y. N. Sun. A general Fatou lemma, *Adv. Math.* **213** (2007), 741–762.

[13] Podczeck, K. Markets with infinitely many commodities and a continuum of agents with non-convex preferences, *Econom. Theory* **9** (1997), 385–426.

[14] Podczeck, K. On the convexity and compactness of the integral of a Banach space valued correspondence, *J. Math. Econom.* **44** (2008), 836–852.

[15] Rustichini, A. A counterexample and an exact version of Fatou's lemma in infinite dimensional spaces, *Arch. Math.* **52** (1989), 357–362.

[16] Rustichini, A. and N. C. Yannelis. What is perfect competition?, in: M. A. Khan and N. C. Yannelis (eds.), *Equilibrium Theory in Infinite Dimensional Spaces*, Springer, Berlin, pp. 249–265, 1991.

[17] Sagara, N. Relaxation and purification for nonconvex variational problems in dual Banach spaces: The minimization principle in saturated measure spaces, forthcoming in *SIAM J. Control Optim.*

[18] Schmeidler, D. Fatou's lemma in several dimensions, *Proc. Amer. Soc.* **24** (1970), 300–306.

[19] Sun, Y. N. Integration of correspondences on Loeb spaces, *Trans. Amer. Math. Soc.* **349** (1997), 129–153.

[20] Sun, Y. N., N. C. Yannelis. Saturation and the integration of Banach valued correspondences, *J. Math. Econom.* **44** (2008), 861–865.

[21] Yannelis, N. C. Fatou's lemma in infinite-dimensional spaces, *Proc. Amer. Math. Soc.* **102** (1988), 303–310.