An order theoretical characterisation of JB-algebras

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Let A be a unital C^*-algebra and equip the self-adjoint part A_{sa} part with the product

$$a \bullet b := \tfrac{1}{2}(ab + ba).$$

The norm satisfies:

$$\|a \bullet b\| \leq \|a\| \, \|b\| , \quad \left\|a^2\right\| = \|a\|^2 , \quad \left\|a^2\right\| \leq \left\|a^2 + b^2\right\| .$$

This is the canonical example of a JB-algebra.

Definition

A *JB-algebra* A is a real Banach space with a commutative (not necessarily associative) bilinear product $a \bullet b$ such that

$$a^2 \bullet (a \bullet b) = a \bullet (a^2 \bullet b) \qquad (\textit{Jordan identity})$$

and the norm satisfies

$$\|a \bullet b\| \leq \|a\| \, \|b\| \, , \quad \left\|a^2\right\| = \|a\|^2 \, , \quad \left\|a^2\right\| \leq \left\|a^2 + b^2\right\|$$

- We will only consider unital JB-algebras in this talk.

Remark

The squares form a closed cone with non empty interior.

- Finite dimensional JB-algebras were completely classified by Jordan, von Neumann, and Wigner.

Every finite dimensional JB-algebra is a direct sum of simple ones:

$$M_n(\mathbb{R})_{sa},\ M_n(\mathbb{C})_{sa},\ M_n(\mathbb{H})_{sa},\ M_3(\mathbb{O})_{sa},\ H \oplus \mathbb{R}$$

Theorem (Koecher-Vinberg)

Let A be a finite dimensional real Hilbert space with a closed cone C having non empty interior. Then A is a JB-algebra for some norm with cone of squares C iff C is symmetric.

Symmetric:

- (self-dual) $\{a \in A \colon \langle a, b \rangle \geq 0 \ \forall\, b \in C\} = C$
- (homogeneous) $\mathrm{Aut}(C) := \{T \in \mathrm{GL}(A) \colon T(C) = C\}$ acts transitively on C°. (for all $a, b \in C^\circ$ there is a $T \in \mathrm{Aut}(C)$ such that $Ta = b$)

Example

Let $A = M_n(\mathbb{R})_{sa},\ M_n(\mathbb{C})_{sa}$, with $\langle M, N\rangle = \mathrm{trace}(MN)$.
We have

$$C = \{M\colon M \text{ is pos. semi-def.}\},\ C^\circ = \{M\colon M \text{ is pos. def.}\}$$

- (self-dual) $\mathrm{trace}(MN) \geq 0$ for all $N \in C$ iff $M \in C$
- (homogeneous) for $Q \in C^\circ$, $M \mapsto Q^{-1/2}MQ^{-1/2}$ is in $\mathrm{Aut}(C)$

Question

Can we generalise the Koecher-Vinberg theorem to infinite dimensions?

- problem: infinite dimensional JB-algebras are generally not Hilbert spaces and therefore cannot have a self-dual cone.

Theorem (Walsh)

Let A be a finite dimensional real Hilbert space with a closed cone C having non empty interior. Then A is a JB-algebra for some norm with cone of squares C iff there is an antitone map $f : C^\circ \to C^\circ$.

Antitone:

- f is a bijection, $a \leq b \Leftrightarrow f(b) \leq f(a)$ and $f(\lambda a) = \lambda^{-1} f(a)$ for all $\lambda > 0$.

Remark

For JB-algebras the inversion map $a \mapsto a^{-1}$ is antitone.

Example (Idea)

For $A = M_n(\mathbb{R})_{sa}$, $M_n(\mathbb{C})_{sa}$ the map $M \mapsto M^{-1}$ is antitone.

$$\begin{aligned} M \leq N &\Leftrightarrow N^{-1/2}MN^{-1/2} \leq I_n \\ &\Leftrightarrow I_n \leq (N^{-1/2}MN^{-1/2})^{-1} = N^{1/2}M^{-1}N^{1/2} \\ &\Leftrightarrow N^{-1} \leq M^{-1} \end{aligned}$$

Finite dimensional real Hilbert spaces A with a closed cone C having non empty interior are *order unit spaces*:

- C is Archimedean
- $u \in C$ is an order unit: for all $a \in A$ there is a $\lambda > 0$ such that $a \leq \lambda u$
- we can equip A with the order unit norm:

$$\|a\|_u := \inf \{\lambda > 0 \colon -\lambda u \leq a \leq \lambda u\}$$

Remark

JB-algebras with their cone of squares and its unit are order unit spaces and the JB-norm coincides with the order unit norm.

Conjecture

Let (A, C, u) be a complete order unit space. Then A is a JB-algebra with cone of squares C iff there is an antitone map $f\colon C^\circ \to C^\circ$.

Definition

Let H be a real Hilbert space with $\dim H \geq 2$ and consider $H \oplus \mathbb{R}$ with product $(x, \lambda) \bullet (y, \mu) := (\mu x + \lambda y, \langle x, y \rangle + \lambda \mu)$ and norm $\|(x, \lambda)\| := \sqrt{\langle x, x \rangle} + |\lambda|$. This JB-algebra is called a *spin factor*.

Remark

$C = \{\lambda(x, 1) \colon \lambda \geq 0,\ x \in B_H\}$, so C is strictly convex.

Theorem (Lemmens, v. Imhoff, R)

Let (A, C, u) be a complete order unit space with strictly convex cone. Then A is a spin factor with cone of squares C iff there is an antitone map $f\colon C^\circ \to C^\circ$.

Theorem (Lemmens, v. Imhoff, R)

Let (A, C, u) be a complete order unit space with strictly convex cone. Then A is a spin factor with cone of squares C iff there is an antitone map $f\colon C^\circ \to C^\circ$.

Thank you for your attention!