REMOVABILITY RESULTS FOR SUBHARMONIC FUNCTIONS, FOR HARMONIC FUNCTIONS AND FOR HOLOMORPHIC FUNCTIONS

JUHANI RIIHENTAUS

University of Oulu, Department of Mathematical Sciences,
P.O. Box 3000, FI-90014 Oulun yliopisto, Finland

and

University of Eastern Finland, Department of Physics and Mathematics,
P.O. Box 111, FI-80101 Joensuu, Finland

riihentaus@member.ams.org and juhani.riihentaus@uef.fi

ABSTRACT. We begin with an improvement to an extension result for subharmonic functions of Blanchet et al. With the aid of this improvement we then give extension results both for harmonic and for holomorphic functions. Our results for holomorphic functions are related to Besicovitch's and Shiffman's well-known extension results, at least in some sense. Moreover, we recall another, slightly related and previous extension result for holomorphic functions.

1. INTRODUCTION

1.1. **An outline.** We will consider extension problems for subharmonic, harmonic and holomorphic functions. Our results are based on an extension result for subharmonic functions, see Theorem 1 in Section 2 below. The starting point for this result is a result of Blanchet. As a matter of fact, Blanchet has

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shown that hypersurfaces of class $\mathcal{C}^1$ are removable singularities for subharmonic functions, provided the considered subharmonic functions satisfy certain assumptions. We have showed that, in certain cases, it is sufficient that the exceptional sets are of finite (n-1)-dimensional Hausdorff measure, see [25], Theorem, p. 568.

In Sections 3 and 4 we will then apply our subharmonic function result to get extension results both for harmonic and for holomorphic functions.

1.2. **Notation.** Our notation is more or less standard, see [23, 24, 25, 26, 27]. However, for the convenience of the reader we recall here the following. We use the common convention $0 \cdot \pm\infty = 0$. For each $n \geq 1$ we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. In integrals we will write dx for the Lebesgue measure in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $0 \leq \alpha \leq n$ and $A \subset \mathbb{R}^n$, $n \geq 1$. Then we write $\mathcal{H}^\alpha(A)$ for the α-dimensional Hausdorff (outer) measure of A. Recall that $\mathcal{H}^0(A)$ is the number of points of A. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $j \in \mathbb{N}$, $1 \leq j \leq n$, then we write $x = (x_j, X_j)$, where $X_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Moreover, if $A \subset \mathbb{R}^n$, $1 \leq j \leq n$, and $x_j^0 \in \mathbb{R}$, $X_j^0 \in \mathbb{R}^{n-1}$, we write

$$A(x_j^0) = \{X_j \in \mathbb{R}^{n-1} : x = (x_j^0, X_j) \in A\}, \; A(X_j^0) = \{x_j \in \mathbb{R} : x = (x_j, X_j^0) \in A\}.$$

If $\Omega \subset \mathbb{R}^n$ and $p > 0$, then $\mathcal{L}^p_{\text{loc}}(\Omega)$, $p > 0$, is the space of functions u in Ω for which $|u|^p$ is locally integrable on Ω.

For the definition and properties of harmonic and subharmonic functions, see e.g. [9, 10, 11, 17, 19], for the definition and properties of holomorphic functions see e.g. [4, 12, 14, 28].

2. Extension results for subharmonic functions

2.1. **A result of Federer.** The following important result of Federer on geometric measure theory will be used repeatedly.

Lemma. ([6], Theorem 2.10.25, p. 188, and [28], Corollary 4, Lemma 2, p. 114) *Suppose that $E \subset \mathbb{R}^n$, $n \geq 2$.*

1. *If $\mathcal{H}^{n-1}(E) = 0$, then for all j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ the set $E(X_j)$ is empty.*
2. *If $\mathcal{H}^{n-1}(E) < +\infty$, then for all j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ the set $E(X_j)$ is finite.*

2.2. **A result of Blanchet.** Blanchet has given the following result:

Blanchet's theorem. ([2], Theorems 3.1, 3.2 and 3.3, pp. 312-313) *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let S be a hypersurface of class $\mathcal{C}^1$ which divides Ω into two subdomains Ω_1 and Ω_2. Let $u \in \mathcal{C}^0(\Omega) \cap \mathcal{C}^2(\Omega_1 \cup \Omega_2)$ be subharmonic* *(respectively convex (or respectively plurisubharmonic provided Ω is then a domain in $\mathbb{C}^n$, $n \geq 1$)) in Ω_1 and Ω_2. If $u_i = u|\Omega_i \in \mathcal{C}^1(\Omega_i \cup S)$, $i = 1,2$, and*

$$\frac{\partial u_i}{\partial \overline{n}^k} \geq \frac{\partial u_k}{\partial \overline{n}^k} \tag{1}$$

on S with $i,k = 1,2$, then u is subharmonic (respectively convex (or respectively plurisubharmonic)) in Ω.

Above $\overline{n}^k = (\overline{n}^k_1, \overline{n}^k_2, \ldots, \overline{n}^k_n)$ is the unit normal exterior to Ω_k, and $u_k \in \mathcal{C}^1(\Omega_k \cup S)$, $k = 1,2$, means that there exist n functions v^j_k, $j = 1,2,\ldots,n$, continuous on $\Omega_k \cup S$, such that

$$v^j_k(x) = \frac{\partial u_k}{\partial x_j}(x)$$

for all $x \in \Omega_k$, $k = 1,2$ and $j = 1,2,\ldots,n$.

The following example shows that one cannot drop the above condition (1) in Blanchet's theorem.

Example. The function $u : \mathbb{R}^2 \to \mathbb{R}$,

$$u(z) = u(x+iy) = u(x,y) := \begin{cases} 1+x, & \text{when } x < 0, \\ 1-x, & \text{when } x \geq 0, \end{cases}$$

is continuous in $\mathbb{R}^2$ and subharmonic, even harmonic in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. It is easy to see that u does not satisfy the condition (1) on $S = \{0\} \times \mathbb{R}$ and that u is not subharmonic in $\mathbb{R}^2$.

Remark. For related results, previous and later, see Khabibullin's results [15], Lemma 2.2, p. 201, Fundamental Theorem 2.1, pp. 200-201, and [16], Lemma 4.1, p. 503, Theorem 2.1, p. 498, Theorems 3.1 and 3.2, pp. 500-501. In this connection, see also [9], 1.4.3, pp. 21-22.

2.3. **An improvement to the result of Blanchet.** Already in [23], Theorem 4, pp. 181-182, we have given partial improvements to the cited subharmonic removability results of Blanchet. For more recent improvements, see [25], Theorem, p. 568, and [27], Theorem 1, p. 154. Now we improve these recent results slightly still more, see Theorem 1 below. Instead of hypersurfaces of class $\mathcal{C}^1$, we consider again arbitrary sets of finite $(n-1)$-dimensional Hausdorff measure as exceptional sets. Then, however, the condition (1) is replaced by another, related condition, the condition (iv) below, which is now, at least seemingly, less stringent as before.

Our result is:

Theorem 1. *Suppose that* Ω *is a domain in* $\mathbb{R}^n$, $n \geq 2$. *Let* $E \subset \Omega$ *be closed in* Ω *and let* $\mathcal{H}^{n-1}(E) < +\infty$. *Let* $u : \Omega \setminus E \to \mathbb{R}$ *be subharmonic and such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.
(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$.
(iii) *For each* j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.
(iv) *For each* j, $1 \leq j \leq n$, *and for* $\mathcal{H}^{n-1}$*-almost all* $X_j \in \mathbb{R}^{n-1}$ *such that* $E(X_j)$ *is finite, the following condition holds:*
For each $x_j^0 \in E(X_j)$ *there exist sequences* $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, *such that* $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ *as* $l \to +\infty$, *and*

(iv(a)) $\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,1}^{0,2}, X_j) \in \mathbb{R}$,
(iv(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then u *has a subharmonic extension to* Ω.

Proof. It is sufficient to show that

$$\int u(x)\, \Delta\varphi(x)\, dx \geq 0$$

for all nonnegative testfunctions $\varphi \in \mathcal{D}(\Omega)$.

Take $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, arbitrarily. Let $K = \operatorname{spt}\varphi$. Choose a domain Ω_1 such that $K \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$ and $\overline{\Omega}_1$ is compact. Since $u \in \mathcal{C}^2(\Omega \setminus E)$ and u is subharmonic in $\Omega \setminus E$, $\Delta u(x) \geq 0$ for all $x \in \Omega \setminus E$. Thus the claim follows if we show that

$$\int u(x)\, \Delta\varphi(x)\, dx \geq \int \Delta u(x)\, \varphi(x)\, dx.$$

For this purpose fix j, $1 \leq j \leq n$, arbitrarily for a while. By Fubini's theorem, see e.g. [5], Theorem 1, pp. 22-23,

$$\int u(x)\, \frac{\partial^2 \varphi}{\partial x_j^2}(x)\, dx = \int \big[\int u(x_j, X_j) \frac{\partial^2 \varphi}{\partial x_j^2}(x_j, X_j)\, dx_j \big]\, dX_j.$$

Using the above Lemma, assumptions (i), (ii) and (iii), and Fubini's theorem, we see that for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$,

$$
(2) \qquad \begin{cases} u(\cdot\,, X_j) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega(X_j)), \\ \dfrac{\partial^2 u}{\partial x_j^2}(\,\cdot\,, X_j) \in \mathcal{L}^1_{\mathrm{loc}}(\Omega(X_j)), \\ E(X_j) \ \text{ is finite, thus there exists } M = M(X_j) \in \mathbb{N} \text{ such that} \\ E(X_j) = \{x_j^1, x_j^2, \ldots, x_j^M\} \text{ where } x_j^k < x_j^{k+1}, k = 1, 2, \ldots, M-1. \end{cases}
$$

Let $X_j \in \mathbb{R}^{n-1}$ be arbitrary as above in (2). We may suppose that $\Omega(X_j)$ is a finite interval. Choose for each $k = 1, 2, \ldots, M$ numbers $a_k, b_k \in (\Omega \setminus E)(X_j)$ such that $a_k < x_j^k < b_k$, $k = 1, 2, \ldots, M$, $a_{k+1} = b_k$, $k = 1, 2, \ldots, M-1$, and that $a_1, b_M \in (\Omega \setminus \overline{\Omega}_1)(X_j)$.

With the aid of (iv) we find for each $x_j^k \in E(X_j)$ sequences $x_{j,l}^{k,l}, x_{j,l}^{k,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, for which

(a) $x_{j,l}^{k,1} \nearrow x_j^k$, $x_{j,l}^{k,2} \searrow x_j^k$ as $l \to +\infty$, and

$$
\lim_{l \to +\infty} u(x_{j,l}^{k,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{k,2}, X_j) \in \mathbb{R},
$$

(b) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{k,1}, X_j) \leq \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{k,2}, X_j) < +\infty$.

Take k, $1 \leq k \leq M$, arbitrarily and consider the interval (a_k, b_k), where $a_k < x_j^k < b_k$. To simplify the notation, write $a := a_k$, $b := b_k$ and $x_j^0 := x_j^k$. Then

$$
a < x_{j,l}^{0,1} \nearrow x_j^0, \; b > x_{j,l}^{0,2} \searrow x_j^0 \text{ as } l \to +\infty.
$$

Then just partial integration!

$$
\begin{aligned}
\int_a^b u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j &= \int_a^{x_j^0} u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j+\\
&+\int_{x_j^0}^b u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j\\
&= \lim_{l\to+\infty}\int_a^{x_{j,l}^{0,1}} u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j + \lim_{l\to+\infty}\int_{x_{j,l}^{0,2}}^b u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j\\
&= \lim_{l\to+\infty}\left[\Big|_a^{x_{j,l}^{0,1}} u(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j) - \int_a^{x_{j,l}^{0,1}} \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j\right]+\\
&+\lim_{l\to+\infty}\left[\Big|_{x_{j,l}^{0,2}}^b u(x_j,X_j)\frac{\partial\varphi}{\partial x_j}(x_j,X_j) - \int_{x_{j,l}^{0,2}}^b \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j\right]\\
&= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right]+\\
&+\lim_{l\to+\infty}\left[u(x_{j,l}^{0,1},X_j)\,\frac{\partial\varphi}{\partial x_j}(x_{j,l}^{0,1},X_j) - \int_a^{x_{j,l}^{0,1}} \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j\right]+\\
&-\lim_{l\to+\infty}\left[u(x_{j,l}^{0,2},X_j)\,\frac{\partial\varphi}{\partial x_j}(x_{j,l}^{0,2},X_j) + \int_{x_{j,l}^{0,2}}^b \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j\right]\\
&= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right]+\\
&-\lim_{l\to+\infty}\int_a^{x_{j,l}^{0,1}} \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j+\\
&-\lim_{l\to+\infty}\int_{x_{j,l}^{0,2}}^b \frac{\partial u}{\partial x_j}(x_j,X_j)\,\frac{\partial\varphi}{\partial x_j}(x_j,X_j)\,dx_j\\
&= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right]+\\
&-\lim_{l\to+\infty}\left[\Big|_a^{x_{j,l}^{0,1}} \frac{\partial u}{\partial x_j}(x_j,X_j)\varphi(x_j,X_j) - \int_a^{x_{j,l}^{0,1}} \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j\right]+\\
&-\lim_{l\to+\infty}\left[\Big|_{x_{j,l}^{0,2}}^b \frac{\partial u}{\partial x_j}(x_j,X_j)\varphi(x_j,X_j) - \int_{x_{j,l}^{0,2}}^b \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j\right]
\end{aligned}
$$

$$= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right] +$$
$$+ \left[\frac{\partial u}{\partial x_j}(a,X_j)\varphi(a,X_j) - \frac{\partial u}{\partial x_j}(b,X_j)\varphi(b,X_j)\right] +$$
$$- \lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1},X_j)\varphi(x_{j,l}^{0,1},X_j)\right] + \lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2},X_j)\varphi(x_{j,l}^{0,2},X_j)\right] +$$
$$+ \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j$$
$$= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right] +$$
$$+ \left[\frac{\partial u}{\partial x_j}(a,X_j)\varphi(a,X_j) - \frac{\partial u}{\partial x_j}(b,X_j)\varphi(b,X_j)\right] +$$
$$+ \left[\lim_{l\to+\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2},X_j) - \lim_{l\to+\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1},X_j)\right]\varphi(x_j^0,X_j) +$$
$$+ \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j$$
$$= \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right] +$$
$$+ \left[\frac{\partial u}{\partial x_j}(a,X_j)\varphi(a,X_j) - \frac{\partial u}{\partial x_j}(b,X_j)\varphi(b,X_j)\right] +$$
$$+ \left[\limsup_{l\to+\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2},X_j) - \liminf_{l\to+\infty}\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1},X_j)\right]\varphi(x_j^0,X_j) +$$
$$+ \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j$$
$$\geq \left[u(b,X_j)\frac{\partial\varphi}{\partial x_j}(b,X_j) - u(a,X_j)\frac{\partial\varphi}{\partial x_j}(a,X_j)\right] +$$
$$+ \left[\frac{\partial u}{\partial x_j}(a,X_j)\varphi(a,X_j) - \frac{\partial u}{\partial x_j}(b,X_j)\varphi(b,X_j)\right] + \int_a^b \frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j.$$

Above we have used just standard properties of limits and our assumption (iv(b)). Observe here, for example, that already the assumptions (i), (ii), (iii) and (iv(a)) imply the existence of the limits

$$\lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1},X_j)\varphi(x_{j,l}^{0,1},X_j)\right] \text{ and } \lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2},X_j)\varphi(x_{j,l}^{0,2},X_j)\right].$$

And as is easily seen, also the limits

$$\lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1},X_j)\right] \text{ and } \lim_{l\to+\infty}\left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2},X_j)\right]$$

exist in the case when $\varphi(x_j^0,X_j)\neq 0$.

To return to our original notation, we have thus obtained for each $k=1,2,\dots,M$,

$$\int_{a_k}^{b_k} u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j \geq \left[u(b_k,X_j)\frac{\partial\varphi}{\partial x_j}(b_k,X_j)-u(a_k,X_j)\frac{\partial\varphi}{\partial x_j}(a_k,X_j)\right]+$$
$$+\left[\frac{\partial u}{\partial x_j}(a_k,X_j)\varphi(a_k,X_j)-\frac{\partial u}{\partial x_j}(b_k,X_j)\varphi(b_k,X_j)\right]+\int_{a_k}^{b_k}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j.$$

Then just to sum over k:

$$\int u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j = \int_{a_1}^{b_M} u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j =$$
$$=\sum_{k=1}^{M}\int_{a_k}^{b_k} u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j \geq$$
$$\geq \sum_{k=1}^{M}\left[u(b_k,X_j)\frac{\partial\varphi}{\partial x_j}(b_k,X_j)-u(a_k,X_j)\frac{\partial\varphi}{\partial x_j}(a_k,X_j)\right]+$$
$$+\sum_{k=1}^{M}\left[\frac{\partial u}{\partial x_j}(a_k,X_j)\varphi(a_k,X_j)-\frac{\partial u}{\partial x_j}(b_k,X_j)\varphi(b_k,X_j)\right]+$$
$$+\sum_{k=1}^{M}\int_{a_k}^{b_k}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j =$$
$$=\int_{a_1}^{b_M}\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j = \int\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\,\varphi(x_j,X_j)\,dx_j.$$

Above we have used the choice of the numbers a_k, b_k, $k=1,2,\dots,M$, and the fact that $a_1, b_M\in(\Omega\setminus\overline{\Omega}_1)(X_j)$.

Integrate then with respect to X_j and use again Fubini's theorem:

$$\int u(x)\frac{\partial^2\varphi}{\partial x_j^2}(x)\,dx = \int\left[\int u(x_j,X_j)\,\frac{\partial^2\varphi}{\partial x_j^2}(x_j,X_j)\,dx_j\right]dX_j \geq$$
$$\geq \int\left[\int\frac{\partial^2 u}{\partial x_j^2}(x_j,X_j)\varphi(x_j,X_j)\,dx_j\right]dX_j = \int\frac{\partial^2 u}{\partial x_j^2}(x)\varphi(x)\,dx.$$

Summing then over $j = 1, 2, \ldots, n$ gives the desired inequality

$$\int u(x)\Delta\varphi(x)\,dx = \int u(x) \sum_{j=1}^{n} \frac{\partial^2\varphi}{\partial x_j^2}(x)\,dx \geq$$

$$\geq \int \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x)\varphi(x)\,dx = \int \Delta u(x)\varphi(x)\,dx \geq 0,$$

concluding the proof. □

Corollary 1. ([25], Theorem, p. 568, and [27], Theorem 1, p. 154) *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \setminus E \to \mathbb{R}$ be subharmonic and such that the following conditions hold:*

(i) $u \in \mathcal{L}^1_{\text{loc}}(\Omega)$.
(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$.
(iii) *For each j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.*
(iv) *For each j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds:*
For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ as $l \to +\infty$, and
(iv(a)) $\lim_{l\to+\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l\to+\infty} u(x_{j,1}^{0,2}, X_j) \in \mathbb{R}$,
(iv(b)) $-\infty < \lim_{l\to+\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \lim_{l\to+\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then u has a subharmonic extension to Ω.

Corollary 2. ([23], Corollary 4A, pp. 185-186) *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \to \mathbb{R}$ be such that*

(i) $u \in \mathcal{C}^1(\Omega)$,
(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$,
(iii) *for each j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$,*
(iv) *u is subharmonic in $\Omega \setminus E$.*

Then u is subharmonic.

Corollary 3. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) = 0$. Let $u : \Omega \setminus E \to \mathbb{R}$ be subharmonic and such that the following conditions hold:*

(i) $u \in \mathcal{L}^1_{\text{loc}}(\Omega)$,
(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$,

(iii) *for each* j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then u *has a subharmonic extension to* Ω.

Proof. Follows directly from Theorem 1 and Federer's above Lemma. □

3. EXTENSION RESULTS FOR HARMONIC FUNCTIONS

3.1. For removability results for harmonic functions see, among others, [7, 8, 18, 29].

Now, using our Theorem 1, we give the following extension result for harmonic functions:

Theorem 2. *Suppose that* Ω *is a domain in* $\mathbb{R}^n$, $n \geq 2$. *Let* $E \subset \Omega$ *be closed in* Ω *and let* $\mathcal{H}^{n-1}(E) < +\infty$. *Let* $u : \Omega \setminus E \to \mathbb{R}$ *be harmonic and such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(ii) *For each* j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(iii) *For each* j, $1 \leq j \leq n$, *and for* $\mathcal{H}^{n-1}$*-almost all* $X_j \in \mathbb{R}^{n-1}$ *such that* $E(X_j)$ *is finite, the following condition holds:*
For each $x_j^0 \in E(X_j)$ *there exist sequences* $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, *such that* $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ *as* $l \to +\infty$, *and*

(iii(a)) $\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,1}^{0,2}, X_j) \in \mathbb{R}$,

(iii(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) = \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then u *has a harmonic extension to* Ω.

Proof. Since the assumptions of Theorem 1 do hold for the subharmonic function u, u has a subharmonic extension u^* to Ω. On the other hand, the assumptions of Theorem 1 hold also for the subharmonic function $v = -u$. Thus $v = -u$ has a subharmonic extension $v^* = (-u)^*$ to Ω. Observe here that, as pointed out in the proof of Theorem 1, also now the limits

$$\lim_{l \to +\infty} \left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \right] \quad \text{and} \quad \lim_{l \to +\infty} \left[\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) \right]$$

exist in the case when $\varphi(x_j^0, X_j) \neq 0$.

Since $-v^* = u^*$, the extension u^* of u is both subharmonic and superharmonic, thus harmonic and the claim follows. □

3.2. Then a concise special case to our above Theorem 2:

Corollary 4. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) = 0$. Let $u : \Omega \setminus E \to \mathbb{R}$ be harmonic and such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$,

(ii) *for each* j, $1 \leq j \leq n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then u has a harmonic extension to Ω.

Proof. With the aid of the above Lemma one sees easily that the assumptions of Theorem 2 are satisfied. □

4. Extension results for holomorphic functions

4.1. Below we give certain counterparts to two of Shiffman's well-known extension results for holomorphic functions. For these Shiffman's results, see, among others [28, 7, 8, 18].

4.2. First a counterpart to the following result:

Shiffman's theorem. ([28], Lemma 3, p. 115, and [8], Theorem 1.1 (b), p. 703) *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) < +\infty$. If $f : \Omega \to \mathbb{C}$ is continuous and $f|\Omega \setminus E$ is holomorphic, then f is holomorphic in Ω.*

Shiffman's proof was based on coordinate rotation, on the use of Cauchy integral formula and on the cited result of Federer, the above Lemma.

For slightly more general versions of Shiffman's result with different proofs, see [20], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [21], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

4.3. Using again here our above Theorem 1, or more directly Theorem 2, we get the following counterpart to Shiffman's above result:

Theorem 3. *Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) < +\infty$. Let $f = u + iv : \Omega \setminus E \to \mathbb{C}$ be holomorphic and such that the following conditions are satisfied:*

(i) $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(ii) *For each* j, $1 \leq j \leq 2n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ *and* $\frac{\partial^2 v}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

(iii) *For each* j, $1 \leq j \leq 2n$, *and for $\mathcal{H}^{2n-1}$-almost all $X_j \in \mathbb{R}^{2n-1}$ such that $E(X_j)$ is finite, the following condition holds:*

For each $x_j^0 \in E(X_j)$ *there exist sequences* $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, *such that* $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ *as* $l \to +\infty$, *and*

(iii(a)) $\lim_{l \to +\infty} f(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} f(x_{j,l}^{0,2}, X_j) \in \mathbb{C}$,

(iii(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) = \limsup_{l \to +\infty}$, *and* $\frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$ *and* $-\infty < \liminf_{l \to +\infty} \frac{\partial v}{\partial x_j}(x_{j,l}^{0,1}, X_j) = \limsup_{l \to +\infty} \frac{\partial v}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then f *has a holomorphic extension to* Ω.

Proof. Write $f = u + iv$. It is sufficient to show that u and v have harmonic extensions u^* and v^* to Ω. As a matter of fact, then $f^* = u^* + iv^* : \Omega \to \mathbb{C}$ is $\mathcal{C}^\infty$ and thus a continuous function. Therefore the claim follows from Shiffman's theorem or also from [20, 21].

Another possibility for the proof is just to observe that the in $\Omega \setminus E$ harmonic functions u and v have by Theorem 2 harmonic extensions u^* and v^* to Ω. Since u^* and v^* are thus $\mathcal{C}^\infty$ functions, the holomorphy of the extension $f^* = u^* + iv^*$ in Ω follows easily. □

Corollary 5. *Suppose that* Ω *is a domain in* $\mathbb{C}^n$, $n \geq 1$. *Let* $E \subset \Omega$ *be closed in* Ω *and let* $\mathcal{H}^{2n-1}(E) = 0$. *Let* $f : \Omega \setminus E \to \mathbb{C}$ *be holomorphic and such that the following conditions are satisfied:*

(i) $f \in \mathcal{L}^1_{\text{loc}}(\Omega)$,

(ii) *for each* j, $1 \leq j \leq 2n$, $\frac{\partial^2 f}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.

Then f *has a holomorphic extension to* Ω.

4.4. Observe that Corollary 5 gives a counterpart to the following result of Shiffman, at least in some sense.

Another theorem of Shiffman. ([28], Lemma 3, p. 115, and [8], Theorem 1.1 (c), p. 703) *Let* Ω *be a domain in* $\mathbb{C}^n$, $n \geq 1$. *Let* $E \subset \Omega$ *be closed in* Ω *and let* $\mathcal{H}^{2n-1}(E) = 0$. *If* $f : \Omega \setminus E \to \mathbb{C}$ *is holomorphic and bounded, then* f *has a unique holomorphic extension to* Ω.

Shiffman's proof was based on coordinate rotation, on the use of Cauchy integral formula, on the already stated important result of Federer, the Lemma above, and on the following classical result of Besicovitch:

Besicovitch's theorem. ([1], Theorem 1, p. 2) *Let* D *be a domain in* $\mathbb{C}$. *Let* $E \subset D$ *be closed in* D *and let* $\mathcal{H}^1(E) = 0$. *If* $f : D \setminus E \to \mathbb{C}$ *is holomorphic and bounded, then* f *has a unique holomorphic extension to* D.

For slightly more general versions of Shiffman's result with different proofs, see [20], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [21], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

4.5. A previous, slightly related result. Observe that, in addition to Corollary 5, also the following result holds:

Theorem 4. *Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic. If for each j, $1 \leq j \leq 2n$, $\frac{\partial f}{\partial x_j} \in \mathcal{L}^2_{\mathrm{loc}}(\Omega)$, then f has a holomorphic extension to Ω.*

The proof follows at once from the following, rather old result:

Proposition. ([13], Corollary 3.6, p. 301) *Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic. If for some $p \in \mathbb{R}$,*

$$\int_{\Omega \setminus E} |f(z)|^{p-2} \sum_{j=1}^{n} |\frac{\partial f}{\partial z_j}(z)|^2 \, dm_{2n}(z) < +\infty,$$

then f has a meromorphic extension f^* to Ω. If $p \geq 0$, then f^* is holomorphic.

For related, partly previous and partly more general results, see [3], Theorem, p. 284, [13], Theorem 3.5, pp. 300-301, and [22], Theorem 3.1, pp. 925-926.

References

[1] Besicovitch, A.S.: On sufficient conditions for a function to be analytic, and on behavior of analytic functions in the neighborhood of non-isolated singular point. Proc. London Math. Soc. (2) **32**, 1-9 (1931)

[2] Blanchet, P.: On removable singularities of subharmonic and plurisubharmonic functions. Complex Variables **26**, 311-322 (1995)

[3] Cegrell, U.: Removable singularity sets for analytic functions having modulus with bounded Laplace mass. Proc. Amer. Math. Soc. **88**, 283-286 (1983)

[4] Chirka, E.M.: Complex Analytic Sets. Kluwer Academic Publisher, Dordrecht (1989)

[5] Evans, Lawrence C., Gariepy, Ronald F.: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, CRC Press, Inc,, Boca Raton (1992)

[6] Federer, H.: Geometric measure theory, Springer, Berlin (1969)

[7] Harvey, R., Polking J.: Removable singularities of solutions of linear partial differential equations. Acta Math. **125**, 39-56 (1970)

[8] Harvey, R., Polking, J.: Extending analytic objects. Comm. Pure and Appl. Mathematics **28**, 701-727 (1975)

[9] Hayman, W.K., Kennedy, P.B.: Subharmonic Functions, Vol. I. Academic Press, London (1976)

[10] Helms, L.L.: Introduction to potential theory. Wiley-Interscience, New York (1969)

[11] Hervé, M.: Analytic and plurisubharmonic functions in finite and infinite dimensional spaces. Lecture Notes in Mathematics, vol. 198. Springer, Berlin (1971)

[12] Jarnicki, M., Pflug, P.: Extension of Holomorphic Functions. Walter de Gruyter, Berlin (2000)

[13] Hyvönen, Jaakko, Riihentaus, Juhani: Removable singularities for holomorphic functions with locally finite Riesz mass. J. London Math. Soc. (2) **35**, 296-302 (1987)

[14] Jarnicki, M., Pflug, P.: Separately Analytic Functions. European Mathematical Society, Zürich (2011)

[15] Khabibullin, B.N.: A uniqueness theorem for subharmonic functions of finite order. Mat. Sb. **182**(6), 811-827 (1991); English transl. in Math USSR Sbornik **73**(1), 195-210 (1992)

[16] Khabibullin, B.N.: Completeness of systems of entire functions in spaces of holomorphic functions. Mat. Zametki **66**(4), 603-616 (1999); English transl. in Math. Notes **66**(4), 495-506 (1999)

[17] Lelong, P.: Plurisubharmonic functions and positive differential forms. Gordon and Breach, New York (1969)

[18] Polking, J.: A survey of removable singularities. Seminar on nonlinear partial differential equations, Berkeley, California, 1983. In: Math. Sci. Res. Inst. Publ (ed. S.S. Chern) **2**, 261-292 (1984), Springer, Berlin (1984)

[19] Rado, T.: Subharmonic functions. Springer, Berlin (1937)

[20] Riihentaus, J.: Removable singularities of analytic functions of several complex variables. Math. Z. **32**, 45-54 (1978)

[21] Riihentaus, J.: Removable singularities of analytic and meromorphic functions of several complex variables. Colloquium on Complex Analysis, Joensuu, Finland, August 24-27, 1978 (Complex Analysis, Joensuu 1978). In: Proceedings (eds. Ilpo Laine, Olli Lehto, Tuomas Sorvali), Lecture Notes in Mathematics, vol. 747, 329-342 (1978), Springer, Berlin (1979)

[22] Riihentaus, Juhani: A nullset for normal functions in several variables. Proc. Amer. Math. Soc. **110**(4), 923-933 (1990)

[23] Riihentaus, J.: Subharmonic functions, mean value inequality, boundary behavior, nonintegrability and exceptional sets. Workshop on Potential Theory and Free Boundary Flows; August 19-27, 2003: Kiev, Ukraine. In: Transactions of the Institute of Mathematics of the National Academy of Sciences of Ukraine, Kiev **1**(3), 169-191 (2004)

[24] Riihentaus, J.: An inequality type condition for quasinearly subharmonic functions and applications. Positivity VII, Leiden, July 22-26, 2013, Zaanen Centennial Conference. In: Ordered Structures and Applications: Positivity VII, Trends in Mathematics, 395-414, Springer International Publishing (2016)

[25] Riihentaus, J.: Exceptional sets for subharmonic functions. J. Basic & Applied Sciences **11**, 567-571 (2015)

[26] Riihentaus, J.: A removability result for holomorphic functions of several complex variables. J. Basic & Applied Sciences **12**, 50-52 (2016)

[27] Riihentaus, J.: Removability results for subharmonic functions, for harmonic functions and for holomorphic functions. Matematychni Studii, **46**(2), 152-158 (2017)

[28] Shiffman, B.: On the removal of singularities of analytic sets. Michigan Math. J. **15**, 111-120 (1968)

[29] Ullrich, D.C.: Removable sets for harmonic functions. Michigan Math. J. **38**, 467-473 (1991)