Burkholder inequalities in Riesz spaces.

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Motivation

We are interested in one of the major inequalities in Martingale Theory, *viz*., the classical Burkholder's inequality . We recall some of the relevant ideas.

Let $\{(X_n, \mathcal{F}_n) : n \geq 1\}$ be a martingale. Martingale increments are given by

$$\Delta X_1 = X_1 \quad \text{and} \quad \Delta X_n = X_n - X_{n-1} \text{ for all } n = 2, 3, ...$$

and the Quadratic Variation Process is defined by

$$S_n(X) = (\Delta X_1)^2 + \cdots + (\Delta X_n)^2 \text{ for all } n = 1, 2, ...$$

Roughly speaking, the Burkholder inequality stipulates that, as far as L^p-norms are concerned, $S_n^{1/2}$ and X_n increase at the same rate. More precisely, for every $p \in (1, \infty)$ there do exist positive real numbers a_p and b_p such that

$$a_p \left\| S_n^{1/2} \right\|_p \leq \|X_n\|_p \leq b_p \left\| S_n^{1/2} \right\|_p .$$

Notations

Let E be a Dedekind complete Riesz space with a distinguished weak unit $e > 0$. Following Kuo, Labuschagne, and Watson [5]

Definition

We call a linear operator T on E a *conditional expectation* if the following conditions are fulfilled.

1. $Te = e$.
2. T is a projection,
3. T is order continuous,
4. T is *strictly positive* (i.e., $Tx > 0$ whenever $x > 0$),
5. The range $R(T)$ of T is a Dedekind complete Riesz subspace of E.

Notations

- Throughout this talk, T stands for a conditional expectation with natural domain $L^1(T)$.
- $L^1(T)$ is a Dedekind complete Riesz space with a weak order unit $e > 0$ and $Te = e$
- For $p \in (1, \infty)$ and $x \in L^p(T)^+$, we consider the p-power x^p as recently defined by Grobler in [2].
- Following [1], we put

$$L^p(T) = \left\{x \in L^1(T) : |x|^p \in L^1(T)\right\}$$

and

$$N_p(x) = T\left(|x|^p\right)^{1/p} \quad \text{for all} \quad x \in L^p(T).$$

- $L^p(T)$ is a Riesz subspace of $L^1(T)$.

Notations

- A *filtration is* a family $\{T_n : n \geq 1\}$ of conditional expectations with $T_1 = T$ and $T_i T_j = T_j T_i = T_i$ whenever $i \leq j$ [3, Definition 3.1].
- A *martingale* is defined in [3, Definition 3.2] to be a family $\{(x_n, T_n) : n \geq 1\}$ where $\{T_n : n \geq 1\}$ is a filtration such that $T_i(x_j) = x_i$ for all i, j with $i \leq j$.
- Keeping the same notations as previously used in the concrete case, it turns out that there exist positive real numbers a_p and b_p such that

$$a_p N_p\left(S_n^{1/2}\right) \leq N_p(x_n) \leq b_p N_p\left(S_n^{1/2}\right).$$

- The proof of this inequality is very technical in nature. Indeed, it is based upon a generalization of the standard Stopping Time and an integral representation of p-powers

Crucial theorem

The key of the proof is the following theorem

Theorem

Let $p, q \in (1, \infty)$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *and* $0 \leq x, w \in L^p(T)$. *Assume that there are* $\beta \in (1, \infty)$ *and* $c \in (0, \infty)$ *for which*

$$tTP_{(x-\beta te)^+}e \leq cTP_{(x-te)^+}w \quad \text{for all } t \in (0, \infty).$$

Then,

$$N_p(x) \leq cq\beta^p N_p(w).$$

Daniell integral

- It is imperative to built a kind of integral of x^p for $p \in (1, \infty)$ and $x \in L^p(T)^+$. In this regard, we have thought about the Daniell Integral in the sense of Grobler [2].
- We recall some of the relevant ideas.
- Given $x \in E$ and $t \in \mathbb{R}$, we denote by p_t the component of e on the projection band $\left\{(x - te)^+\right\}^d$. In other words, $p_t = e - P_{(x-te)^+}e$.
- The family $(p_t)_{t \in \mathbb{R}}$ is an increasing right continuous system of components of e [4].
- We also put $p_\infty = e$ and $p_{-\infty} = 0$.

Daniell integral

- The Daniell Integral J is then defined on characteristic functions $\chi_{(a,b]}$ by

$$J\left(\chi_{(a,b]}\right) = p_b - p_a \quad \text{for all } a, b \in \mathbb{R} \cup \{\pm\infty\} \quad \text{with } a < b.$$

  Then the definition can be extended in a quite standard way to a large class of functions.

- First, for step functions of the form $\sum_{k=1}^{n} \lambda_k \chi_{]a_k, b_k]}$ via linearity.
- Next, for limit of increasing sequences of positive step functions via order continuity.
- It is customary to denote $J(f)$ by $f(x)$

Riemann integral
Definitions

- A function $f:[a,b]\to E$ is said to be *bounded* if there is $M\in E^{+}$ such that $|f(x)|\leq M \quad$ for all $x\in[a,b]$.
- Let $f:[a,b]\longrightarrow E$ be a bounded function and $\sigma=\{a=x_0<...<x_n=b\}$ a partition of $[a,b]$. The mesh of σ is defined as$\|\sigma\|=\max\{x_i-x_{i-1}:i=1,...,n\}$.
- For $i\in\{1,...,n\}$, we put

$$M_i=M_i(f,\sigma)=\sup\{f(t):x_{i-1}\leq t\leq x_i\}$$

and

$$m_i=m_i(f,\sigma)=\inf\{f(t):x_{i-1}\leq t\leq x_i\}$$

- Define the *upper* and *lower sums* of f with respect to the partition σ by

$$U(f,\sigma)=\sum_{k=1}^{n}M_i(x_i-x_{i-1}) \quad \text{and} \quad L(f,\sigma)=\sum_{k=1}^{n}m_i(x_i-x_{i-1}),$$

Riemann integral

- As in the classical case, we may prove quite easily that for every partitions σ and τ of $[a, b]$, we have $L(f, \sigma) \leq U(f, \tau)$
- Moreover if α and β are tow partitions of $[a, d]$ with β is finer than α then
$$L(f, \alpha) \leq L(f, \beta) \leq U(f, \beta) \leq U(f, \alpha)$$
- .Since E is Dedekind complete, we derive that
$$L(f) = \sup L(f, \sigma) \quad \text{and} \quad U(f) = \inf U(f, \sigma)$$

  exist.

Riemann integral

Definition

Let a, b be two real numbers with $a < b$. A bounded function $f : [a, b] \to E$ is said to be Riemann integrable *if*

$$L(f) = U(f).$$

We write $\int_a^b f(t)\, dt$ (or, briefly, $\int_a^b f$) for the common value.

Riemann integral

It sould be expected that the Riemann integral can be obtained as a limit of certains sequences.
So, if we define a *Riemann sum* of a function $f : [a, b] \longrightarrow E$ with respect to a tagged partition (σ, θ) of $[a, b]$ by

$$S(f, \sigma, \theta) = \sum_{i=1}^{n} f(\theta_i)(x_i - x_{i-1}).$$

Riemann integral

As in classical case we require

Theorem

Let a, b be two real numbers with $a < b$ and $f : [a, b] \longrightarrow E$ be a bounded function.

(i) If there exist two sequences of partitions (α_n) and (β_n) of $[a, b]$ such that $U(f, \beta_n) - L(f, \alpha_n) \longrightarrow 0$, then $f \in \mathrm{RI}([a, b], E)$ and

$$\int_a^b f = \lim_{n \longrightarrow \infty} L(f, \alpha_n) = \lim_{n \longrightarrow \infty} U(f, \beta_n).$$

(ii) If $f \in \mathrm{RI}([a, b], E)$ and if $((\sigma_n, \theta^n))_{n \geq 1}$ is a sequence of tagged partitions of $[a, b]$ with $\|\sigma_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then

$$\lim_{n \longrightarrow 0} S(f, \sigma_n, \theta^n) = \int_a^b f.$$

Riemann integral

Corollary

Let a, b be two real numbers with $a < b$, and $f : [a, b] \longrightarrow E$ be monotone function. Then f is Riemann integrable.

Proof.

Without loss of generality, we may suppose that f is increasing. Let σ_n be a regular partition of $[a, b]$ with $\|\sigma\| = \dfrac{b-a}{n}$. It is sufficent to observe that

$$U(f, \sigma_n) - L(f, \sigma_n) = \frac{b-a}{n}(f(b) - f(a)).$$

□

Riemann integral

We collect here some interesting properties of the integral.

Theorem

Let a, b be real numbers with $a < b$. Then the following hold.

(i) $\mathrm{RI}\left(\left[a,b\right],E\right)$ *is a Riesz space with respect to the pointwise operations and ordering.*

(ii) *The function that takes any* $f \in \mathrm{RI}\left(\left[a,b\right],E\right)$ *to* $\int_a^b f\left(t\right)dt$ *is a positive operator.*

(iii) *If* $f \in \mathrm{RI}\left(\left[a,b\right],E\right)$ *then*

$$\Phi \circ f \in \mathrm{RI}\left(\left[a,b\right],E\right) \quad \textit{and} \quad \int_a^b \Phi \circ f = \Phi \int_a^b f.$$

if either of these conditions is satisfied
a) Φ *is order continuous and lattice homomorphism;*
b) Φ *is order continuous and* f *has bounded variation.*

Integral representation of p-power

Combining a Daniell and Riemann integrals we require

Lemma

Let $p \in (1, \infty)$ and $a, \varepsilon \in (0, \infty)$ with $\varepsilon < a$, then

$$x^p - \varepsilon^p e = \int_\varepsilon^a pt^{p-1} P_{(x-te)^+} e dt \quad \textit{for all } x \in E \textit{ with } \varepsilon e < x \leq ae.$$

Dual formula

Now we will introduce some well-known results in probability theory extended in the framework of Riesz spaces which will be usefull to achieve our objective.
This theorem is a Riesz space version of well-known result in probability theory

Theorem

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \leq x \in L^p(T)$. Then,

$$\begin{aligned} N_p(x) &= \sup\{T(xy) : 0 \leq y \in L^q(T) \text{ and } N_q(y) \leq e\} \\ &= \sup\{T(xy) : 0 \leq y \in L^q(T) \cap L^2(T) \text{ and } N_q(y) \leq e\} \end{aligned}$$

Hölder inequality

Hölder inequality will take the following form.

Theorem

Let T be a conditional expectation with domain $L^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in L^p(T)$ and $y \in L^q(T)$ then

$$xy \in L^1(T) \quad \text{and} \quad N_1(xy) \leq N_p(x)\, N_q(y).$$

Sampling optional theorem

One of the classical properties of stopped submartingales will be extended in the following theorem, this is what we call in litterature sampling optional theorem.

Theorem

Let $P = (P_i)_{i \geq 1}$ *be a stopping time adapted to the filtration* $(T_i)_{i \geq 1}$*.*
Then $T(x_{P \wedge k}) \leq T(x_k)$ *for* $k = 1, 2, \ldots$.

Doob inequality

The following Theorem gives a Riesz spaces version of Doob inequality in classical probability theory.

Theorem

If $t \in (0, \infty)$ *then*

$$tTP_{(M_k - te)^+} e \leq TP_{(M_k - te)^+} x_k .$$

with

$$M_k = \sup_{1 \leq i \leq k} x_i \quad \textit{for all } k \geq 1$$

.

Technical lemmas

As before, we define the quadratic variation by putting

$$S_k = \sum_{j=1}^{k} (\Delta x_j)^2 \quad \text{for all } k \geq 1.$$

Also, we set

$$M_k = \sup_{1 \leq i \leq k} x_i \quad \text{for all } k \geq 1.$$

Lemma

The following holds

$$tTP^d_{(M_n - te)^+} P_{(S_n - t^2 e)^+} e \leq 2Tx_n \quad \textit{for all } t \in (0, \infty).$$

Technical lemmas

Lemma

Let $t \in (0, \infty)$ *and* $c \in [1, \infty)$. *Then*

$$tTP_{(S_n-(2+c)t^2e)^+}P^d_{(M_n-te)^+}e \leq 2TP_{(S_n-ct^2e)^+}x_n.$$

Proof.

Apply the previous result to the positive martingale $(y_i = P_i x_i)_{i\geq 1}$ with $P = (P_i)_{i\geq 1} = (P_{(S_i-ct^2e)^+})_{i\geq 1}$. □

Technical lemmas.

Lemma

Let $c \in [1, \infty)$ *and put*

$$\beta = \sqrt{1 + \frac{2}{c}} \quad \text{and} \quad w = \sup(M_n, (c^{-1} S_n)^{1/2}).$$

Then

$$tTP_{(w-\beta te)^+} e \leq 3TP_{(w-te)^+} x_n \quad \text{for all } t \in (0, \infty).$$

Proof.

Since for $x \in L^2(T)^+$ and $t \in (0, \infty)$ the equality $B_{(x^2 - t^2 e)^+} = B_{(x-te)^+}$ holds.
combining with the previous lemma and Doob inequality introduced before, we obtain the required result. □

Main result

We gathered now all the tools that we need to proove the main result in this talk.

Theorem

For every $p \in (1, \infty)$, there exist constants c_p and C_p such that

$$C_p N_p(x_n) \leq N_p(S_n^{\frac{1}{2}}) \leq c_p N_p(x_n).$$

for all positive martingales $(x_k)_{k \geq 1}$ in $L^2(T) \cap L^p(T)$ with quadratic variation $(S_k)_{k \geq 1}$.

Proof of the right side inequality.

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- Using the crucial theorem, we get

$$N_p(w) \leq 3q\beta^p N_p(x_n),$$

Hence,

$$N_p(S_n^{\frac{1}{2}}) \leq c_p N_p(x_n)$$

with $c_p = 3c^{\frac{1}{2}}q\beta^p$.

□

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- Choose $y \in L^q(T) \cap L^2(T)^+$ with $N_q(y) \leq e$.
- We introduce a new martingale with associate quadratic sum G_n, we use Hölder inequality and Cauchy Shwartz inequality to get

$$T(x_n y) \leq c_q N_p\left(\sqrt{S_n}\right)$$

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- In summary, $T(x_n y) \leq c_q N_p\left(\sqrt{S_n}\right)$ for all y in $L^q(T) \cap L^2(T)^+$ with $N_q(y) \leq e$.
- It follows from Dual formula that $N_p(x_n) \leq c_q N_p\left(\sqrt{S_n}\right)$ .The conclusion would be clear putting $C_p = \dfrac{1}{c_q}$, which completes the proof of the theorem.

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Thank you for your attention