Extending Algebra Homomorphisms to Spectral Measures

(a Boolean Algebra approach)

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K compact Hausdorff space (or locally compact H. sp.)

$C(K)$ real-valued continuous functions (or $C_0(K)$ real-valued cont. fns.that are 0 at ∞)

$\mathbf{\Sigma}$ Borel σ-algebra of K

$B(\mathbf{\Sigma})$ bounded real-valued Borel measurable functions on K

X Banach sp., $L(X)$ bounded operators on X

E Banach lattice, $L_r(E)$ regular operators on E

$m : A \to L(X)$ bounded algebra homomorphism,

where A ($C(K)$ or $C_0(K)$)

Definition 1 *A map* $\phi : \mathbf{\Sigma} \to L(X)$ *is a* **spectral measure** *if it is a Boolean algebra homomorphism. Then* ϕ *is* **countably additive** *if it is countably additive in the strong operator topology on* $L(X)$

That is for $u, v \in \mathbf{\Sigma}$, one has $\phi(\chi_u \chi_v) = \phi(\chi_u)\phi(\chi_v)$ and $\phi(\chi_{u \cup v}) = \phi(\chi_u) + \phi(\chi_v) - \phi(\chi_u \chi_v)$

where χ_u is the characteristic function of $u \in \mathbf{\Sigma}$.

Problem 2 *Given $m : A \to L(X)$, what are the necessary and sufficient conditions such that there exists $\phi : B(\mathbf{\Sigma}) \to L(X)$ (bounded algebra homomorphism) with (i) $\phi|_A = m$ and (ii) $\phi|_{\mathbf{\Sigma}}$ is countably additive?*

Problem 3 *Consider the same problem when $L(X)$ is replaced by $L_r(E)$ with the further requirement that m and ϕ should be* **positive**.

Problem 4 *Which Banach lattices E have affirmative answer to Problem 3 for every positive (bounded algebra homomorphism) $m : A \to L_r(E)$?*

Given $m : A \to L(X)$, define a bilinear map

$A \times X \to X$: $(a, x) \rightsquigarrow m(a)(x) = ax$

Consider its (First-)Arens extension ($a \in A, x \in X, x' \in X', x'' \in X'', a'' \in A''$)

$$(1)\ x' \cdot a(x) = x'(ax),\ \text{with } x' \cdot a \in X'$$
$$(2)\ \psi_{x'',x'}(a) = x''(x' \cdot a),\ \text{with } \psi_{x'',x'} \in A'$$
$$(3)\ a'' \cdot x''(x') = a''(\psi_{x'',x'}),\ \text{with } a'' \cdot x'' \in X''$$

Define $\widehat{m} : A'' \to L(X'') : \widehat{m}(a'')(x'') = a'' \cdot x''$

then $\widehat{m}$ is a (w*,w*-operator)-continuous bounded algebra homomorphism and $\widehat{m}(a) = m''(a)$

Lemma 5 *TFAE:*
1. $\widehat{m}(A'')(X) \subset X$,
2. For each $x \in X$, the map $A \to X$ defined by $a \to ax$ is weakly compact,
3. m has a unique extension to a bounded algebra homomorphism $\widehat{m}_X : A'' \to L(X)$ that is (w,weak-operator)-cont.*
($\widehat{m}_X(a'') = \widehat{m}(a'')|_X$).

We refer to the property in condition (2) above as A has weakly compact action on X.

Theorem 6 *Let $m : A \to L(X)$ be a bounded algebra homomorphism. Then there exists a countably additive spectral measure $\phi : B(\mathbf{\Sigma}) \to L(X)$ that extends m if and only if m induces weakly compact action of A in X. In such a case $\phi = \widehat{m}|_{B(\mathbf{\Sigma})}$.*

Corollary 7 *Let $m : A \to L_r(E)$ be a positive bounded algebra homomorphism. Then $\widehat{m}$ is also positive. Furter-more there exists a positive countably additive spectral measure $\phi : B(\mathbf{\Sigma}) \to L_r(E)$ that extends m if and only if m induces weakly compact action of A in E. In such a case $\phi = \widehat{m}|_{B(\mathbf{\Sigma})}$.*

Now we consider Problem 4. However the answer splits into two parts. To state the results we need to recall a definition.

Definition 8 *(Wickstead) The center $Z(E)$ of a Banach lattice is* **topologically full** *if for each positive $x \in E$, the norm closure of $Z(E)x$ is an order ideal in E.*

Theorem 9 *Suppose E has a topologically full center. Then every positive bounded algebra homomorphism $m : C(K) \to L_r(E)$ extends to a positive countably additive spectral measure if and only if E has order continuous norm.*

But when we bring in homomorphisms on $C_0(K)$ as well the answer changes.

Theorem 10 *Suppose E has a topologically full center. Then every positive bounded algebra homomorphism $m : A \to L_r(E)$ extends to a positive countably additive spectral measure if and only if E is a KB-space.*

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