Large sublattices in subsets of Banach lattices

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Main question

Question

Suppose A is a "large" subset of a Banach lattice X. Does $A \cup \{0\}$ contain large (closed) sublatices?

Convention: All spaces, lattices etc. are infinite dimensional, unless specified otherwise.

"Large" may mean that a sublattice is:

- Infinite dimensional.
- Dense in $A \cup \{0\}$.
- Has "many" generators, in the lattice sense (not in the topological sense). If S is a minimal set of generators of Z, S' is another set of generators, and S is infinite, then $|S| \leqslant |S'|$.

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Some history: Banach space case

Definition (Lineability and spaceability)

For a Banach space X, $A \subset X$ is:

- Lineable if $A \cup \{0\}$ contains a linear subspace.
- Spaceable if $A \cup \{0\}$ contains a closed linear subspace.
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Let $ND[0,1]$ be the space of nowhere differentiable functions in $C[0,1]$.

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Suppose X is a Banach lattice. A subset $A \subset X$ is (completely) latticeable if X contains a (closed) infinite dimensional sublattice Z so that $Z \subset A \cup \{0\}$.

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Complements of closed subspaces

Suppose Y is a closed subspace of a Banach lattice X. What kind of sublattices does $(X \setminus Y) \cup \{0\}$ contain?

Theorem

(a) If Y is a closed subspace of a Banach lattice X with $\dim X/Y \geqslant n \in \mathbb{N}$, then $\exists$ an n-dimensional sublattice $Z \subset X$ so that $Z \cap Y = \{0\}$.
(b) Consequently, if Y is a closed subspace of a Banach lattice X with $\dim X/Y = \infty$, then $\forall n \in \mathbb{N}$ $\exists$ an n-dimensional sublattice $Z \subset X$ s.t. $Z \cap Y = \{0\}$.

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In (b), can Z be infinite dimensional?

Complements of closed ideals

An subspace Y of a Banach lattice X is called an ideal if, for any $y \in Y$, and any $x \in X$ satisfying $|x| \leqslant |y|$, we have $x \in Y$.

Theorem

Suppose Y is a closed ideal in X, with $\dim X/Y = \infty$. Then X_+ contains disjoint non-zero elements $(x_i)_{i \in \mathbb{N}}$ so that $Y \cap Z = \{0\}$, where $Z = \overline{\mathrm{span}}[(x_i)_{i \in \mathbb{N}}]$. In particular, $X \backslash Y$ is completely latticeable.

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Suppose Y is a fin. dim. subspace of a Banach lattice X. Then X_+ contains disjoint non-zero elements $(x_i)_{i\in\mathbb{N}}$ so that $Y \cap Z = \{0\}$, where Z is the closed ideal generated $(x_i)_{i\in\mathbb{N}}$:

$$Z = \left\{ z \in X : |z| \leqslant |x| \text{ for some } x \in \overline{\mathrm{span}}[(x_i)_{i\in\mathbb{N}}] \right\}$$

The result of this theorem is sharp.

Proposition

$C[0,1]$ contains a closed inf. codim. subspace Y so that $(C[0,1]\setminus Y) \cup \{0\}$ contains no non-trivial ideals.

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Complements of closed inf. codim. subspaces

Theorem

Suppose X is an infinite dimensional order continuous Banach lattices, and $Y \subset X$ is a closed infinite codimensional subspace. Then $X \setminus Y$ is completely latticeable.

Definition

X is order continuous if, for any net $x_\alpha \searrow 0$, we have $\lim_\alpha \|x_\alpha\| = 0$. **Examples:** L_p $(1 \leqslant p < \infty)$, c_0, but not $C[0,1]$.

Theorem

Suppose K is a compact subset of $\mathbb{R}^n$, $K_0 \subset K$, $X = \{x \in C(K) : x|_{K_0} = 0\}$. If $Y \subset X$ is a closed infinite codimensional subspace, then $X \setminus Y$ is completely latticeable.

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Complements of closed subspaces in atomic lattices

Suppose X is a sequence space – that is, the order structure is determined by a 1-unconditional basis $(\sigma_i)_{i\in\mathbb{N}}$.

Theorem

Suppose Y is a closed subspace of X, with $\dim X/Y = \infty$. Then there exist $k_1 < k_2 < \dots$ so that

$$Y \cap \overline{\mathrm{span}}\left[(\sigma_{k_i})_{i\in\mathbb{N}}\right] = \{0\}.$$

In particular, $X\backslash Y$ is completely latticeable.

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Complements of closed subspaces in ℓ_p, c_0

Proposition

Suppose X is either ℓ_p $(1 < p < \infty)$ or c_0, and Y is a closed subspace of X, with $\dim X/Y = \infty$. Then $X \backslash Y$ is completely latticeable. Moreover, there exists a closed sublattice $Z \subset X$ and a constant c so that $\|z + y\| \geqslant c\|z\|$ for any $z \in Z$ and $y \in Y$.

Remark

Proposition fails for $X = \ell_1$.

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Complements of compact sets

L. Drewnowski 1984 (generalized by D. Kitson and R. Timoney 2011): if A is a non-closed operator range in X, then $X \backslash A$ is spaceable.

Theorem

Suppose A is a relatively compact set in an infinite dimensional Banach lattice X. Then $X \backslash \mathbb{R}A$ is completely latticeable.

Corollary

If X is an infinite dimensional Banach lattice, and $Y \subset X$ is the range of a compact operator, then $X \backslash Y$ is completely latticeable.

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Complements of dense subspaces

Theorem

For $0 < p \leqslant \infty$, there exists a vector lattice $Z \subset \ell_p \backslash (\cup_{q<p} \ell_q) \cup \{0\}$ (or $Z \subset c_0 \backslash (\cup_{q<\infty} \ell_q) \cup \{0\}$ if $p = \infty$) so that:

1. $\overline{Z} = \ell_p$ *(c_0 if $p = \infty$).*
2. *If a set S generates Z as a vector lattice, then $|S| \geqslant 2^{\aleph_0}$.*

Proof: complements of inf. codim. subspaces

Theorem (to be proved)

Suppose X is an infinite dimensional order continuous Banach lattice, and $Y \subset X$ is a closed infinite codimensional subspace. Then $X \backslash Y$ is completely latticeable.

Lemma (Technical lemma, see Singer, Bases II)

Suppose Y is a closed subspace of a Banach lattice X, and $\exists$ mutually disjoint $x_1, x_2, \ldots \in X_+ \backslash \{0\}$ s.t. $Y \cap \mathrm{span}[x_1, x_2, \ldots] = \{0\}$. Then $\exists$ $i_1 < i_2 < \ldots$ with $Y \cap \overline{\mathrm{span}}[x_{i_1}, x_{i_2}, \ldots] = \{0\}$. Consequently, $X \backslash Y$ is completely latticeable.

Strategy for proving Theorem: find mutually disjoint $x_1, x_2, \ldots \in X_+ \backslash \{0\}$ s.t. $Y \cap \mathrm{span}[x_1, x_2, \ldots] = \{0\}$.

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Suppose X is an infinite dimensional order continuous Banach lattice, and $Y \subset X$ is a closed infinite codimensional subspace. Then $X \backslash Y$ is completely latticeable.

It suffices to consider the situation when X has a weak order unit. Then X is a Köthe function space on (Ω, Σ, μ), where μ is a σ-finite measure on the measure space (Ω, Σ).

For simplicity, we assume that μ is an atomless probability measure.

Notation. For $Z \subset X$, and $S \in \Sigma$, set $Z_S = \{z \in Z : z|_{S^c} = 0\}$.

Lemma

For X, Y, S as above, either $\dim X_S / Y_S = \infty$, or $\dim X_{S^c} / Y_{S^c} = \infty$.

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Sketch of proof. Find a family of sets $U_t \in \Sigma$ ($t \in [0,1]$) s.t.

- $U_0 = \emptyset$, $U_1 = \Omega$, and $\mu(U_t) = t$ for any t.
- If $t < s$, then $U_t \subset U_s$.

Set $\phi(t) = \dim X_{U_t}/Y_{U_t}$. ϕ is increasing, left continuous. Thus, $\exists \alpha \in [0,1)$ s.t. $\{t \in [0,1] : \dim X_{U_t}/Y_{U_t} \geqslant n\} = (\alpha, 1]$. Similarly, $\exists \beta \in (0,1]$ s.t. $\{t \in [0,1] : \dim X_{U_t^c}/Y_{U_t^c} \geqslant n\} = [0, \beta)$.

We have $\max\{\dim X_{U_t}/Y_{U_t}, \dim X_{U_t^c}/Y_{U_t^c}\} = \infty$, hence $[0,\beta) \cup (\alpha, 1] = [0,1]$. Pick $t \in (\beta, \alpha)$, and take either $S = U_t$, or $S = U_{t^c}$.

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We have $\max\{\dim X_{U_t} / Y_{U_t}, \dim X_{U_t^c} / Y_{U_t^c}\} = \infty$, hence $[0, \beta) \cup (\alpha, 1] = [0, 1]$. Pick $t \in (\beta, \alpha)$, and take either $S = U_t$, or $S = U_{t^c}$.

Proof: complements of inf. codim. subspaces

Lemma

$\forall\ n \in \mathbb{N}\ \exists\ S \in \Sigma$ s.t. $\dim X_S/Y_S \geqslant n$, and $\dim X_{S^c}/Y_{S^c} = \infty$.

Sketch of proof. Find a family of sets $U_t \in \Sigma$ $(t \in [0,1])$ s.t.

- $U_0 = \emptyset$, $U_1 = \Omega$, and $\mu(U_t) = t$ for any t.
- If $t < s$, then $U_t \subset U_s$.

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Theorem (to be proved)

Suppose X is an infinite dimensional order continuous Banach lattice, and $Y \subset X$ is a closed infinite codimensional subspace. Then $X \backslash Y$ is completely latticeable.

Sketch of proof. Assume X is a Köthe function space on (Ω, Σ, μ), where μ is an atomless probability measure. Need to find mutually disjoint $x_1, x_2, \ldots \in X_+ \backslash \{0\}$ s.t. $Y \cap \mathrm{span}[x_1, x_2, \ldots] = \{0\}$.

Recursively find $x_1, x_2, \ldots \in X_+ \backslash \{0\}$, and mutually disjoint $S_1, S_2, \ldots \subset \Omega$ so that:

- $\forall i$, x_i is supported on S_i.
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Once $x_1, \ldots, x_n, S_1, \ldots, S_n$ are selected: find $S_{n+1} \subset S = (S_1 \cup \ldots \cup S_n)^c$ s.t. $\dim X_{S_{n+1}} / Y_{S_{n+1}} \geqslant n+1$, $\dim X_{S \setminus S_{n+1}} / Y_{S \setminus S_{n+1}} = \infty$.
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Theorem (to be proved)

If Y is a closed subspace of a Banach lattice X with $\dim X/Y \geqslant n \in \mathbb{N}$, then there exists an n-dimensional sublattice $Z \subset X$ so that $Z \cap Y = \{0\}$.

Definition

A Banach lattice X is Dedekind (or order) complete if any subset of X, which has an upper bound, has a supremum.

Examples of Dedekind complete lattices:
L_p $(1 \leqslant p \leqslant \infty)$, dual Banach lattices.
$C[0,1]$ is not Dedekind complete.

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Proof when X is Dedekind complete.

(1) If G is a subspace of a fin. dim. Banach lattice F, then $\exists$ a sublattice $Z \subset F$, s.t. $Z \cap G = \{0\}$, $\dim Z + \dim G = \dim F$.

Proof: identify F with $\mathbb{R}^m$ ($m = \dim F$), use linear algebra.

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Find a subspace $E \subset X$ s.t. $\dim E = n$, $E \cap Y = \{0\}$. Use (2) to find a fin. dim. sublattice $F \subset X$ and $T \in B(X)$ as above s.t. $TE \cap Y = \{0\}$. Then $\dim F/G \geqslant n$, where $G = Y \cap F$. Use (1) to find a sublattice $Z \subset F$ s.t. $\dim Z = n$, $Z \cap G = \{0\}$. Then $Z \cap Y = \{0\}$. ∎

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Proof for general X. X^{**} is Dedekind complete, hence $\exists$ n-dimensional sublattice $W \subset X^{**}$ s.t. $W \cap Y^{\perp\perp} = \{0\}$.

Find $c \in (0, 1/9)$ s.t. $\operatorname{dist}(w, Y^{\perp\perp}) \geqslant 3c\|w\|$ $\forall\, w \in W$. Find $x_1^*, \dots, x_N^* \in \mathbf{B}(Y^\perp) \subset X^*$ s.t.

$$\max_{1\leqslant i\leqslant N} \left|\langle x_i^*, w\rangle\right| \geqslant 2c\|w\| \quad \forall\, w \in W.$$

Let $V = \left\{x^{**} \in X^{**} : \max_{1\leqslant i\leqslant N} \left|\langle x_i^*, w\rangle\right| < c\right\}$.

Local reflexivity: $\exists$ lattice homomorphism $T : W \to Z \subset X$ s.t. $\|T\|, \|T^{-1}\| < 1 + \varepsilon$, and $(I - T)\mathbf{B}(W) \subset \varepsilon V$. $\Rightarrow Z \cap Y = \{0\}$. ■

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