Applications of the scarcity theorem in ordered Banach algebras

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Outline of Talk

- Background, Motivation and References
- Notation and Preliminaries
- Ordered Banach Algebras
- The Scarcity Theorem
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Theorem (informal version of Aupetit's scarcity theorem)

If a function f is analytic on a domain D in the complex plane and with values in a Banach algebra, then either the subset of D on which the spectrum of f is finite is "very small" in some sense, or it is the whole of D, in which case the spectrum of f is even uniformly finite on D.

Background, Motivation and References

- B. Aupetit: *A Primer on Spectral Theory*. Springer, New York (1991).
- H. du T. Mouton and H. Raubenheimer: On rank one and finite elements of Banach algebras. *Studia Math.* 104 (1993), 211–219.
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- H. du T. Mouton and S. Mouton: Domination properties in ordered Banach algebras. *Studia Math.* 149 (2002), 63–73.

- R. Brits: On the multiplicative spectral characterisation of the Jacobson radical. *Quaest. Math.* 31 (2008), 179–188.
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- G. Braatvedt and R. Brits: Uniqueness and spectral variation in Banach algebras. *Quaest. Math.* 36 (2013), 155–165.
- F. Schulz and R. Brits: Uniqueness under spectral variation in the socle of a Banach algebra. *J. Math. Anal. Appl.* 444 (2016), 1626–1639.
- G. Braatvedt, R. Brits and F. Schulz: Rank in Banach algebras: a generalized Cayley-Hamilton theorem. *Linear Algebra Appl.* 507 (2016), 389–398.

Background, Motivation and References

Solution of a domination problem using the scarcity theorem:

Theorem (H. du T. Mouton and S. Mouton, 2002)

Let A be an ordered Banach algebra with certain natural properties and let a and b be positive elements in A such that b dominates a. If b is in the radical of A, then so is a.

- S. Mouton: Applications of the scarcity theorem in ordered Banach algebras. *Studia Math.* 225 (2014), 219–234.

Notation and Preliminaries

A: complex unital Banach algebra

$\sigma(a)$: the spectrum of $a \in A$

$\sigma'(a)$: the non-zero spectrum of a

$\eta\sigma(a)$: the connected hull of $\sigma(a)$, i.e. $\sigma(a)$ together with its holes

$\#\sigma(a)$: the number of elements in $\sigma(a)$

Notation and Preliminaries

Rad(A): the radical of A

A is *semisimple* if $\mathrm{Rad}(A) = \{0\}$.

$Z(A) = \{a \in A : ax - xa \in \mathrm{Rad}(A) \text{ for all } x \in A\}$: the *center modulo the radical* of A

QN(A): the set of all a with $\sigma(a) = \{0\}$

Notation and Preliminaries

Let A be a semisimple Banach algebra and $a \in A$:

If $a \neq 0$, then a is a *rank one element* if $aAa \subseteq \mathbb{C}a$.

a is a *finite rank element* if $a = 0$ or a is a finite sum of rank one elements.

$\mathrm{Soc}(A)$: the socle of A, i.e. the sum of the minimal left ideals in A

$\mathrm{Soc}(A)$ consists of all finite rank elements.

Notation and Preliminaries

$C(K)$: the Banach algebra of all continuous complex valued functions on a compact Hausdorff space K

$\mathcal{L}(E)$: the Banach algebra of all bounded linear operators on a Banach lattice E

$\mathcal{L}^r(E)$: the Banach algebra of all regular operators on a Banach lattice E

Notation and Preliminaries

Let A be a Banach algebra and D a domain in $\mathbb{C}$. Then $g : A \to A$ is *D-analytic* if $g \circ f : D \to A$ is analytic for every analytic function $f : D \to A$.

The following maps are D-analytic, for every domain $D \subseteq \mathbb{C}$:

- $g(x) = a + x$ and $g(x) = a(1 + x)$ (for fixed $a \in A$)
- every continuous, linear map, e.g. $g(x) = ax$

Let X be a vector space, $a \in X$ and $U \subseteq X$:
Then a is an *absorbing point* of U if for all $x \in X$ there exists $r > 0$ such that $a + \lambda x \in U$ for all real λ with $|\lambda| \leq r$.
U is an *absorbing set* if U contains an absorbing point.

Ordered Banach Algebras

Let A be a complex unital Banach algebra:

A non-empty subset C of A is called a *space cone* if C is closed under addition and under non-negative real scalar multiplication.

C is an *algebra cone* if C is a space cone containing 1 which is closed under multiplication.

A is called an *ordered Banach algebra* (OBA) if A contains an algebra cone C.

A is then partially ordered by C:

$$a \leq b \text{ if and only if } b - a \in C$$

The elements of C are called the *positive elements*.

C is *normal* if there exists a constant $\alpha > 0$ with the following property:
if $0 \leq a \leq b$ (relative to C), then $||a|| \leq \alpha ||b||$.

Proposition (S. Mouton, 2014)

Let A be an OBA with closed and normal algebra cone C. Then C is not an absorbing set.

C is *generating* if $\mathrm{span}(C) = A$.

Example

Let K be a compact Hausdorff space and let C be the subset of $C(K)$ consisting of all functions which are real and nonnegative at every point of K. Then $C(K)$ is an OBA with closed, normal and generating algebra cone C.

Example

Let E be a complex Banach lattice with cone $C = \{x \in E : x = |x|\}$ and let $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$. Then both $\mathcal{L}(E)$ and $\mathcal{L}^r(E)$ are OBAs with closed and normal algebra cone K, and K generates $\mathcal{L}^r(E)$.

The Scarcity Theorem

Theorem (Aupetit's scarcity theorem)

Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite is a Borel set having zero capacity, or there exist an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\sigma(f(\lambda)) = n$ for all $\lambda \in D \backslash E$ and $\#\sigma(f(\lambda)) < n$ for all $\lambda \in E$.

Corollary

Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. If $n \geq 1$ is such that $\#\sigma(f(\lambda)) \leq n$ for all λ in a subset of D with non-zero capacity, then $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in D$.

Applications of the Scarcity Theorem

Theorem (S. Mouton, 2014)

Let A be an OBA with algebra cone C, G a subset of A and B a subset of C which is a space cone of A containing a point which is absorbing in G. Also, let $g : A \to A$ be a $\mathbb{C}$-analytic map.

1. *If $\#\sigma(g(c)) < \infty$ for all $c \in B \cap G$, then there exists $m \in \mathbb{N}$ such that $\#\sigma(g(x)) \leq m$ for all $x \in \mathrm{span}(B)$.*
2. *If $n \in \mathbb{N}$ and $\#\sigma(g(c)) \leq n$ for all $c \in B \cap G$, then $\#\sigma(g(x)) \leq n$ for all $x \in \mathrm{span}(B)$.*
3. *If $\sigma(g(c)) = \{0\}$ for all $c \in B \cap G$, then $\sigma(g(x)) = \{0\}$ for all $x \in \mathrm{span}(B)$.*

Typical choices, depending on the situation:

- $B = B_1 \cap C$ for any vector subspace B_1 of A; in particular $B = C$. For some applications, $B = \mathrm{QN}(A) \cap C$.
- $G = A$ or $G = A^{-1}$ or G a neighborhood of 0 or 1.

Applications of the Scarcity Theorem

Theorem (classical result)

Let A be a Banach algebra. Then $\mathrm{Rad}(A) = \{a \in A : Aa \subseteq \mathrm{QN}(A)\}$.

Theorem

Let A be a Banach algebra and let G be any set with absorbing point 0. *Then* $\mathrm{Rad}(A) = \{a \in A : Ga \subseteq \mathrm{QN}(A)\}$.

Theorem (H. du T. Mouton and S. Mouton, 2002; S. Mouton, 2014)

Let A be an OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G. Then $\mathrm{Rad}(A) = \{a \in A : (C \cap G)a \subseteq \mathrm{QN}(A)\}$.

Applications of the Scarcity Theorem

Theorem (B. Aupetit, 1970s)

Let A be a Banach algebra. If A contains an absorbing subset U such that

1. $\sigma(x)$ *is finite for all* $x \in U$, *then* $A/\mathrm{Rad}(A)$ *is finite-dimensional,*
2. $\#\sigma(x) \leq n$ *for all* $x \in U$ *and some fixed* $n \in \mathbb{N}$, *then* $\dim A/\mathrm{Rad}(A) \leq n^6$.

Theorem (S. Mouton, 2014)

Let A be an OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G.

1. *If* $\#\sigma(c)$ *is finite for all* $c \in C \cap G$, *then* $A/\mathrm{Rad}(A)$ *is finite-dimensional.*
2. *If* $\#\sigma(c) \leq n$ *for all* $c \in C \cap G$ *and some fixed* $n \in \mathbb{N}$, *then* $\dim A/\mathrm{Rad}(A) \leq n^6$.

Applications of the Scarcity Theorem

Theorem (S. Mouton, 2014)

Let A be an OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G.

- *If $\#\sigma(c)$ is finite for all $c \in C \cap G$, then $A/\mathrm{Rad}(A)$ is finite-dimensional.*
- *If $\#\sigma(c) = 1$ for all $c \in C \cap G$, then $A/\mathrm{Rad}(A) \cong \mathbb{C}$.*

Take $G = A^{-1}$:

Corollary

Let A be a semisimple OBA with generating algebra cone C. Then

1. $\dim A < \infty$ *if and only if the spectrum of each positive invertible element in A is finite, and*
2. $A \cong \mathbb{C}$ *if and only if the spectrum of each positive invertible element in A consists of one element only.*

Applications of the Scarcity Theorem

Theorem (B. Aupetit, 1970s)

Let A be a Banach algebra and $a \in A$. If $\#\sigma(ax - xa) = 1$ for all $x \in A$, then $a \in Z(A)$.

Theorem (S. Mouton, 2014)

Let A be an OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G. If $a \in A$ and $\#\sigma(ac - ca) = 1$ for all $c \in C \cap G$, then $a \in Z(A)$.

Applications of the Scarcity Theorem

Theorem (H. du T. Mouton and H. Raubenheimer, 1993; B. Aupetit and H. du T. Mouton, 1994)

Let A be a semisimple Banach algebra. Then

$$\{a \in A : \text{ there exists } n \in \mathbb{N} \text{ such that } \#\sigma'(xa) \leq n \text{ for all } x \in A\}$$

$$= \mathrm{Soc}(A) = \{a \in A : \#\sigma'(xa) < \infty \text{ for all } x \in A\},$$

and if $0 \neq a \in A$, then a is rank one if and only if $\#\sigma'(xa) \leq 1$ for all $x \in A$.

Applications of the Scarcity Theorem

Theorem (S. Mouton, 2014)

Let A be a semisimple OBA with generating algebra cone C, and let G be any subset of A which contains a point of C which is absorbing in G. Then

$$\{a \in A : \text{ there exists } n \in \mathbb{N} \text{ s.t. } \#\sigma'(ca) \leq n \text{ for all } c \in C \cap G\}$$

$$= \mathrm{Soc}(A) = \{a \in A : \#\sigma'(ca) < \infty \text{ for all } c \in C \cap G\},$$

and if $\dim A = \infty$ *and* $0 \neq a \in A$*, then a is rank one if and only if* $\#\sigma'(ca) \leq 1$ *for all* $c \in C \cap G$*.*

Applications of the Scarcity Theorem

Theorem (H. du T. Mouton and H. Raubenheimer, 1993; B. Aupetit and H. du T. Mouton, 1994)

Let A be a semisimple Banach algebra and $a \in A$. Then the following are equivalent:

1. $a \in \mathrm{Soc}(A)$
2. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \backslash \{0\}$ and all $x \in A$.*
3. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F} \eta\sigma(x + ta) \subseteq \eta\sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \backslash \{0\}$ and all $x \in A$.*

Applications of the Scarcity Theorem

Theorem (S. Mouton, 2014)

Let A be a semisimple OBA with closed and generating algebra cone C and $a \in A$. Then the following are equivalent:

1. $a \in \mathrm{Soc}(A)$.
2. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \backslash \{0\}$ and all $x \in C \cap A^{-1}$.*
3. *There exists $n \in \mathbb{N}$ such that $\cap_{t \in F} \eta\sigma(x + ta) \subseteq \eta\sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \backslash \{0\}$ and all $x \in C \cap A^{-1}$.*

Applications of the Scarcity Theorem

Theorem (H. du T. Mouton and H. Raubenheimer, 1993; B. Aupetit and H. du T. Mouton, 1994)

Let A be a semisimple Banach algebra and $0 \neq a \in A$. Then the following are equivalent:

1. *a is rank one.*
2. $\sigma(x + s_0 a) \cap \sigma(x + s_1 a) \subseteq \sigma(x)$ *for all* $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ *with* $s_0 \neq s_1$ *and all* $x \in A$.
3. $\eta\sigma(x + s_0 a) \cap \eta\sigma(x + s_1 a) \subseteq \eta\sigma(x)$ *for all* $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ *with* $s_0 \neq s_1$ *and all* $x \in A$.

Applications of the Scarcity Theorem

Theorem (S. Mouton, 2014)

Let A be a semisimple OBA with $\dim A = \infty$ and closed and generating algebra cone C, and let $0 \neq a \in A$. Then the following are equivalent:

1. *a is rank one.*
2. *$\sigma(c + s_0 a) \cap \sigma(c + s_1 a) \subseteq \sigma(c)$ for all $s_0, s_1 \in \mathbb{C} \backslash \{0\}$ with $s_0 \neq s_1$ and all $c \in C \cap A^{-1}$.*
3. *$\eta\sigma(c + s_0 a) \cap \eta\sigma(c + s_1 a) \subseteq \eta\sigma(c)$ for all $s_0, s_1 \in \mathbb{C} \backslash \{0\}$ with $s_0 \neq s_1$ and all $c \in C \cap A^{-1}$.*

THANK YOU