Lipschitz structure in ordered Banach spaces

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THE CLAUDE LEON FOUNDATION

Question

X a Banach space
$C \subseteq X$ closed generating cone/wedge

Do there exist Lipschitz functions $(\cdot)^{\pm} : X \to C$ so that $x = x^+ - x^-$ for all $x \in X$?

'Lipschitz', meaning there exists $\alpha \geq 0$ so that
$\|x^{\pm} - y^{\pm}\| \leq \alpha \|x - y\| \quad (x, y \in X)$

Motivation

Completions of normed function spaces

Answer

…sometimes

Banach lattices

Answer

...sometimes

Banach lattices
...trivially

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Finite dimensional ordered Banach spaces

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...slightly harder

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Finite dimensional ordered Banach spaces

...slightly harder

...in general?

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Answer

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YES!

There exist continuous Lipschitz functions $(\cdot)^{\pm} : X \to C$
so that $x = x^+ - x^-$ for all $x \in X$

Answer

$$C \oplus C \to X, \quad (a, b) \mapsto a - b$$

Surjective, hence open (Klee 1955; Andô 1962)

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(de Jeu and M. 2014)

Ernest A. Michael

(1925 --- 2013)

Bartle-Graves-like application of Michael's Selection Theorem to

$$X \ni x \mapsto \{(a, b) \in C \oplus C : x = a - b\}$$

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Publications results for "MR Number=(MR0930904)"

MR0930904 (89d:54012) Reviewed
Yost, David(5-ANU)
There can be no Lipschitz version of Michael's selection theorem.
Proceedings of the analysis conference, Singapore 1986, 295–299,
North-Holland Math. Stud., 150, *North-Holland, Amsterdam*, 1988.
54C65 (46B20)
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Lipschitz properties of

$$\varphi : S_X \to 2^{C \oplus C}$$

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There exists $\alpha > 0$, for every $x_0 \in S_X$ and $(a_0, b_0) \in \varphi(x_0)$,

$$\psi : S_X \to 2^{C \oplus C}$$

$$\psi(x) := \varphi(x) \cap ((a_0, b_0) + \alpha \|x - x_0\| B_{X \oplus X}) \neq \emptyset$$

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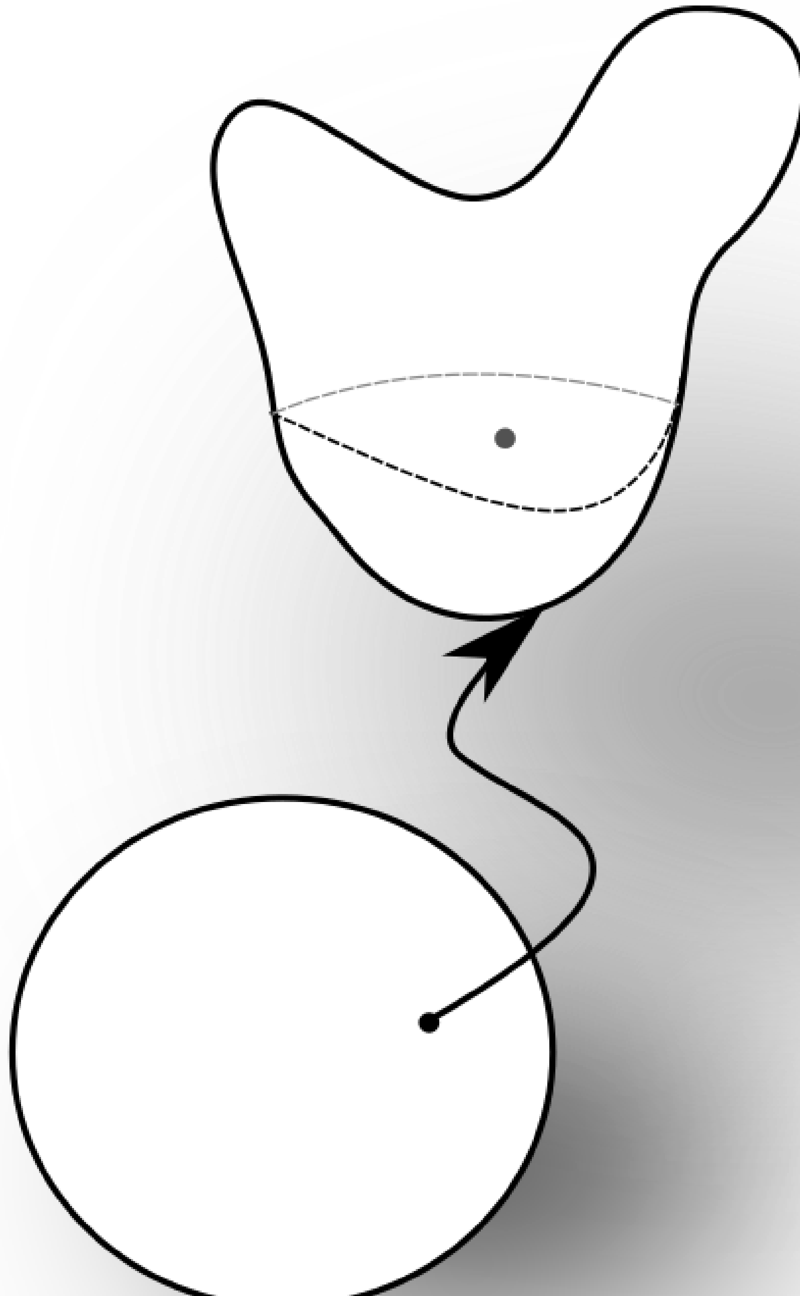
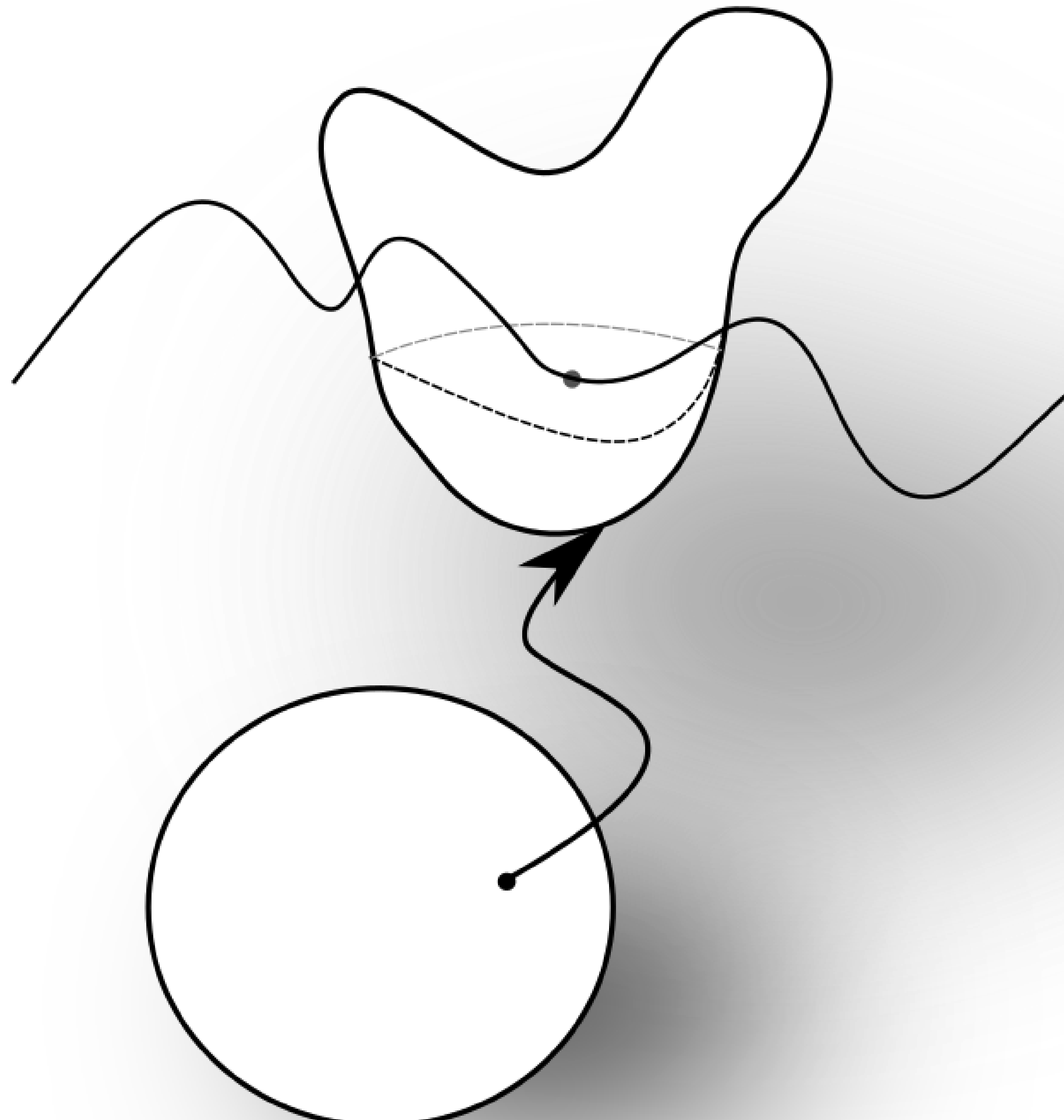
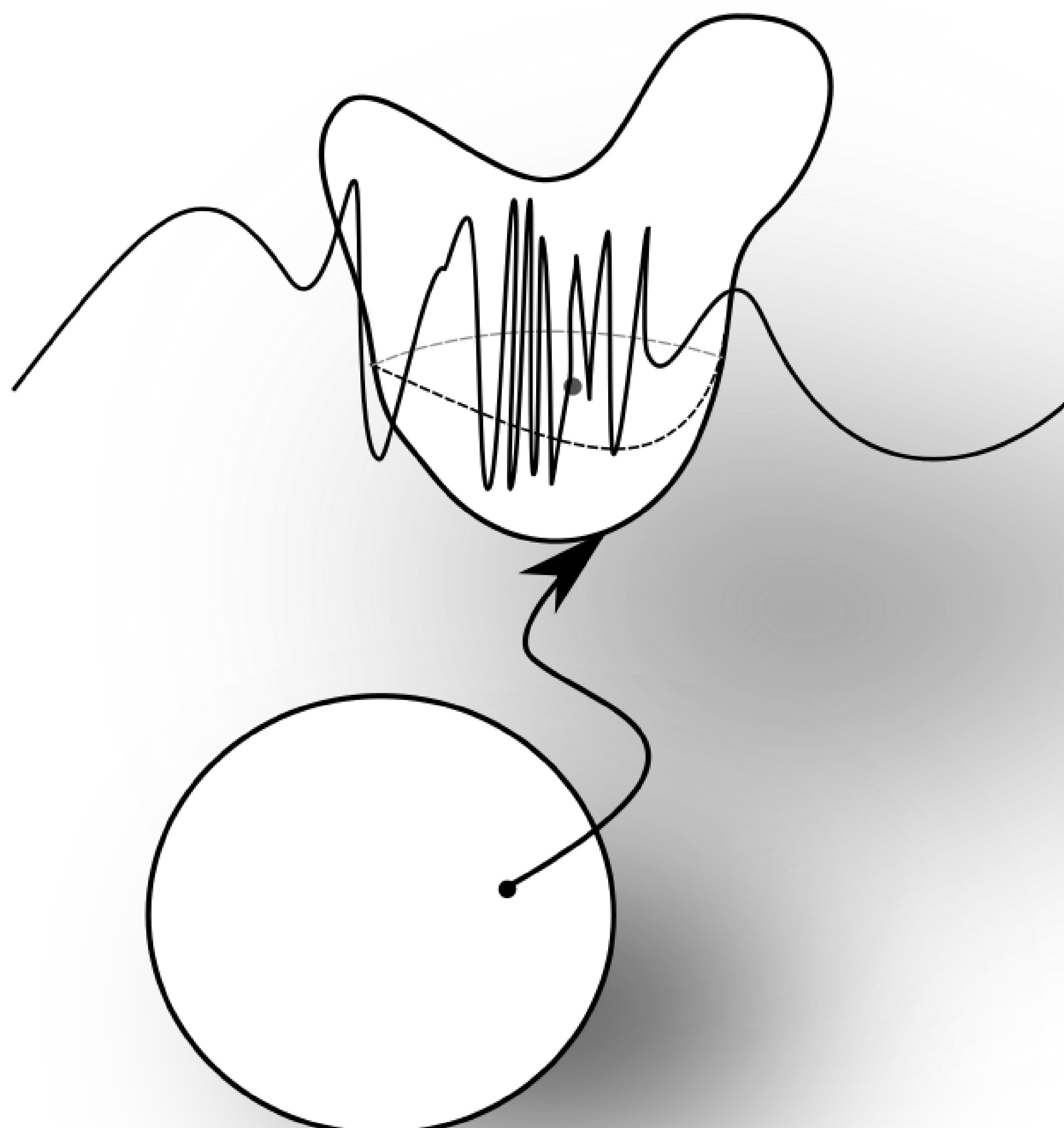
There exists continuous functions $(\cdot)^{\pm} : S_X \to C$
with $x = x^+ - x^-$

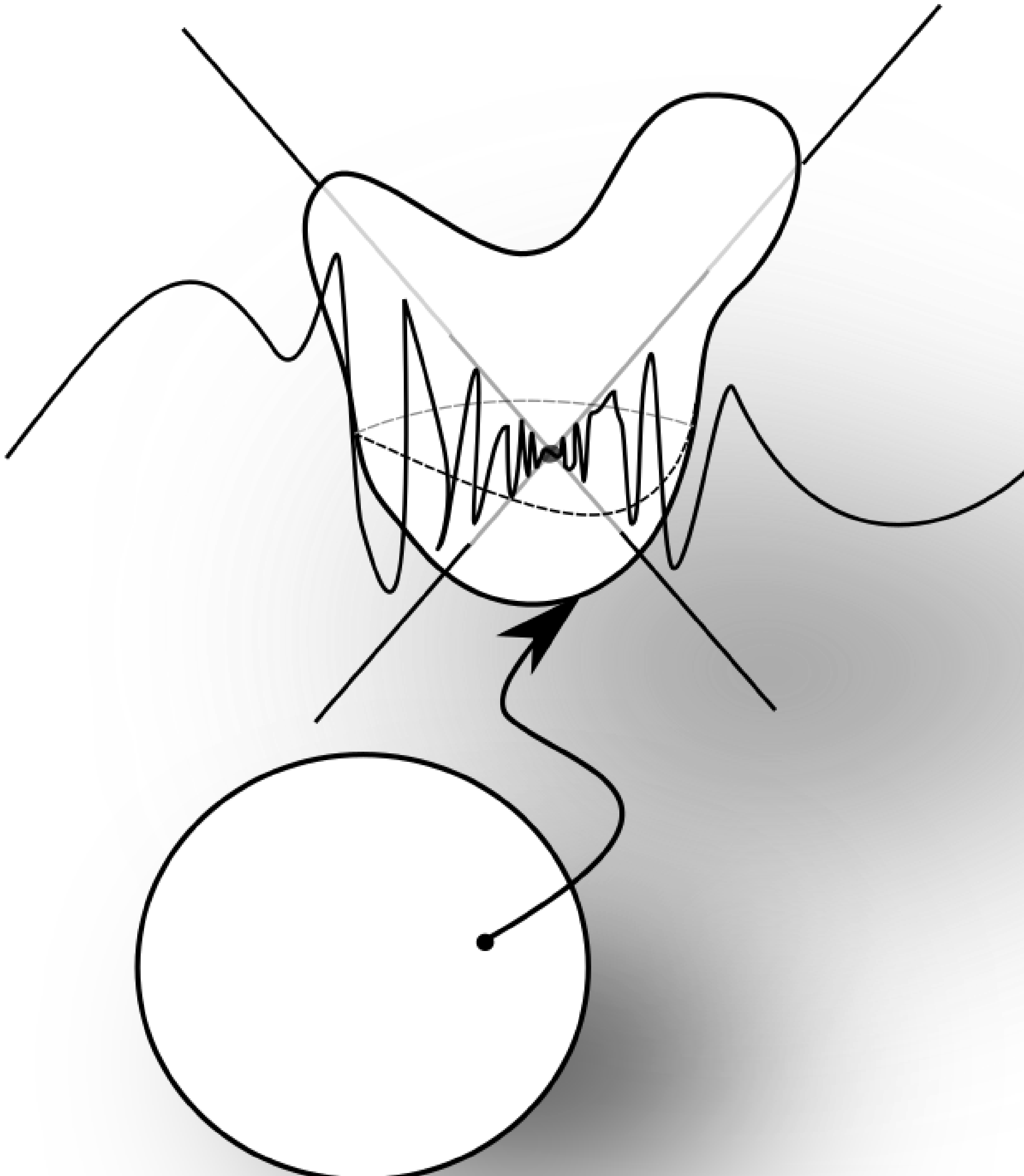
and $\|x_0^{\pm} - x^{\pm}\| \leq \alpha \|x_0 - x\| \qquad (x \in S_X)$

"pointwise Lipschitz at x_0"









Theorem

X a Banach space
$C \subseteq X$ closed generating cone/wedge

There exist Lipschitz continuous functions $(\cdot)^{\pm} : X \to C$ so that $x = x^{+} - x^{-}$ for all $x \in X$,

and both $(\cdot)^{\pm} : X \to C$ are pointwise Lipschitz on a dense set of X.

A Pointwise Lipschitz Selection Theorem (M. 2016 preprint)

M a metric space, Y a Banach space.
$\varphi : M \to 2^Y$ a “nice enough” “Lipschitz-like” multifunction.

There exists a continuous function $f : M \to Y$ that is pointwise Lipschitz on a dense set of M that satisfies $f(x) \in \varphi(x) \quad (x \in M)$

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... and we can do no better than a dense set.

Theorem (Durand-Cartagena, Jaramillo 2010)

M (bi-Lipschitz homeomorphic to) length-metric space,
Y a Banach space.

If $f : M \to Y$ is pointwise Lipschitz everywhere,
then it is Lipschitz.

Example by Lindenstrauss and Aharoni (1978).

Conjecture

There exists a Banach space X,
closed generating cone/wedge $C \subseteq X$,

which cannot have Lipschitz functions
$(\cdot)^{\pm} : X \to C$ with $x = x^{+} - x^{-}$ for all $x \in X$.

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Any ideas?

http://miek.me