Interpolation between L^1 and L^∞, operators of conditional mathematical expectation, and regular functions.

Alexander Mekler

Saint Petersburg's Mathematical Society

July 21, 2017

- We denote the unit interval $I := (0, 1]$ with the Lebesgue σ-algebra Λ and Lebesgue measure λ by (I, λ, Λ).

- We denote the unit interval $I := (0, 1]$ with the Lebesgue σ-algebra Λ and Lebesgue measure λ by (I, λ, Λ).
- As usual, $L^0 := L^0(I, \Lambda, \lambda)$ is the vector lattice of all (classes of equivalency λ-a.e.) real Λ-measurable finite-valued functions with the order induced by the relation $f \leq g$ λ-a.e.

- We denote the unit interval $I := (0,1]$ with the Lebesgue σ-algebra Λ and Lebesgue measure λ by (I, λ, Λ).
- As usual, $L^0 := L^0(I, \Lambda, \lambda)$ is the vector lattice of all (classes of equivalency λ-a.e.) real Λ-measurable finite-valued functions with the order induced by the relation $f \leq g$ λ-a.e.
- The function identically equal to 1 on $(0,1]$ is denoted by $\mathbf{1}$.

- We denote the unit interval $I := (0, 1]$ with the Lebesgue σ-algebra Λ and Lebesgue measure λ by (I, λ, Λ).
- As usual, $L^0 := L^0(I, \Lambda, \lambda)$ is the vector lattice of all (classes of equivalency λ-a.e.) real Λ-measurable finite-valued functions with the order induced by the relation $f \leq g$ λ-a.e.
- The function identically equal to 1 on $(0, 1]$ is denoted by $\mathbf{1}$.
- Also, as usual, by $L^1 := L^1(I, \Lambda, \lambda)$ (respectively, $L^\infty := L^\infty(I, \lambda, \Lambda)$) we denote the subspace of L^0 that consists of all $f \in L^0$ such that $\int\limits_0^1 |f| d\lambda < \infty$ (respectively, the subspace of all λ-essentially bounded functions).

- Two functions $f, g \in L^1$ are called *equimeasurable* if for any real α we have $\lambda(\{f > \alpha\}) = \lambda(\{g > \alpha\})$.

- Two functions $f, g \in L^1$ are called *equimeasurable* if for any real α we have $\lambda(\{f > \alpha\}) = \lambda(\{g > \alpha\})$.
- If $f \in L^0$ then by $f^\star$ is denoted the non-increasing function equimesurable with $|f|$. It is well known and obvious that $f^\star$ exists, is unique up to λ-a.e. equivalence, and that $f \in L^1 \Leftrightarrow f^\star \in L^1$.

- Two functions $f, g \in L^1$ are called *equimeasurable* if for any real α we have $\lambda(\{f > \alpha\}) = \lambda(\{g > \alpha\})$.
- If $f \in L^0$ then by $f^\star$ is denoted the non-increasing function equimesurable with $|f|$. It is well known and obvious that $f^\star$ exists, is unique up to λ-a.e. equivalence, and that $f \in L^1 \Leftrightarrow f^\star \in L^1$.
- If $f \in L^1$ then the function $f^{\star\star}$ is defined in the standard way, namely

$$f^{\star\star}(t) = \frac{1}{t}\int_0^t f^\star(s)ds, \ t \in I.$$

- Two functions $f, g \in L^1$ are called *equimeasurable* if for any real α we have $\lambda(\{f > \alpha\}) = \lambda(\{g > \alpha\})$.
- If $f \in L^0$ then by $f^\star$ is denoted the non-increasing function equimesurable with $|f|$. It is well known and obvious that $f^\star$ exists, is unique up to λ-a.e. equivalence, and that $f \in L^1 \Leftrightarrow f^\star \in L^1$.
- If $f \in L^1$ then the function $f^{\star\star}$ is defined in the standard way, namely

$$f^{\star\star}(t) = \frac{1}{t} \int_0^t f^\star(s)ds, \ t \in I.$$

- Two functions $f, g \in L^1$ are called *equivalent* if for some constant C, $C > 1$, we have

$$\frac{1}{C} < \frac{f}{g} < C \text{ a.e. on } I.$$

- A subset $\mathcal{A}$ of Λ is called a σ-subalgebra if
  (a) $a_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} a_i \in \mathcal{A}$ and $\bigcap_{i=1}^{\infty} a_i \in \mathcal{A}$.
  (b) $a \in \mathcal{A} \Rightarrow (I \setminus a) \in \mathcal{A}$.
  (c) $a \in \Lambda, \lambda(a) = 0 \Rightarrow a \in \mathcal{A}$.

- A subset $\mathcal{A}$ of Λ is called a σ-subalgebra if
  (a) $a_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup\limits_{i=1}^{\infty} a_i \in \mathcal{A}$ and $\bigcap\limits_{i=1}^{\infty} a_i \in \mathcal{A}$.
  (b) $a \in \mathcal{A} \Rightarrow (I \setminus a) \in \mathcal{A}$.
  (c) $a \in \Lambda, \lambda(a) = 0 \Rightarrow a \in \mathcal{A}$.
- The set $a \in \mathcal{A}$ is called an *atom* in $\mathcal{A}$ if $\lambda(a) > 0$ and for any $b \in \mathcal{A}$, such that $b \subseteq a$, either $\lambda(b) = 0$ or $\lambda(a \setminus b) = 0$.

- A subset $\mathcal{A}$ of Λ is called a σ-subalgebra if
  (a) $a_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} a_i \in \mathcal{A}$ and $\bigcap_{i=1}^{\infty} a_i \in \mathcal{A}$.
  (b) $a \in \mathcal{A} \Rightarrow (I \setminus a) \in \mathcal{A}$.
  (c) $a \in \Lambda, \lambda(a) = 0 \Rightarrow a \in \mathcal{A}$.
- The set $a \in \mathcal{A}$ is called an *atom* in $\mathcal{A}$ if $\lambda(a) > 0$ and for any $b \in \mathcal{A}$, such that $b \subseteq a$, either $\lambda(b) = 0$ or $\lambda(a \setminus b) = 0$.
- A σ-subalgebra $\mathcal{A}$ is called *continuous* if it contains no atoms; it is called *discrete* if the union of all its atoms has measure 1.

- A subset $\mathcal{A}$ of Λ is called a σ-subalgebra if
  (a) $a_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} a_i \in \mathcal{A}$ and $\bigcap_{i=1}^{\infty} a_i \in \mathcal{A}$.
  (b) $a \in \mathcal{A} \Rightarrow (I \setminus a) \in \mathcal{A}$.
  (c) $a \in \Lambda, \lambda(a) = 0 \Rightarrow a \in \mathcal{A}$.
- The set $a \in \mathcal{A}$ is called an *atom* in $\mathcal{A}$ if $\lambda(a) > 0$ and for any $b \in \mathcal{A}$, such that $b \subseteq a$, either $\lambda(b) = 0$ or $\lambda(a \setminus b) = 0$.
- A σ-subalgebra $\mathcal{A}$ is called *continuous* if it contains no atoms; it is called *discrete* if the union of all its atoms has measure 1.
- Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-subalgebras of Λ. We say that $\mathcal{H}$ is *finer* than $\mathcal{G}$ if $a \in \mathcal{G} \Rightarrow a \in \mathcal{H}$. Respectively, in this case we will say that $\mathcal{G}$ is *coarser* than $\mathcal{H}$.

- We say that a σ-subalgebra $\mathcal{A}$ is complemented (in Λ) if there is another σ-subalgebra $\mathcal{A}^{\perp}$ (called the *independent complement* of $\mathcal{A}$) such that

- We say that a σ-subalgebra $\mathcal{A}$ is complemented (in Λ) if there is another σ-subalgebra $\mathcal{A}^{\perp}$ (called the *independent complement* of $\mathcal{A}$) such that
- 1. Λ is the smallest σ-subalgebra that is finer than both $\mathcal{A}$ and $\mathcal{A}^{\perp}$.
  2. $\lambda(d \cap e) = \lambda(d)\lambda(e)$ for any $d \in \mathcal{A}$ and $e \in \mathcal{A}^{\perp}$.

- We say that a σ-subalgebra $\mathcal{A}$ is complemented (in Λ) if there is another σ-subalgebra $\mathcal{A}^{\perp}$ (called the *independent complement* of $\mathcal{A}$) such that
- 1. Λ is the smallest σ-subalgebra that is finer than both $\mathcal{A}$ and $\mathcal{A}^{\perp}$.
  2. $\lambda(d \cap e) = \lambda(d)\lambda(e)$ for any $d \in \mathcal{A}$ and $e \in \mathcal{A}^{\perp}$.
- A discrete σ-subalgebra $\mathcal{F}$ is also called an *at most countable partition*.

- We say that a σ-subalgebra $\mathcal{A}$ is complemented (in Λ) if there is another σ-subalgebra $\mathcal{A}^\perp$ (called the *independent complement* of $\mathcal{A}$) such that
-
  1. Λ is the smallest σ-subalgebra that is finer than both $\mathcal{A}$ and $\mathcal{A}^\perp$.
  2. $\lambda(d \cap e) = \lambda(d)\lambda(e)$ for any $d \in \mathcal{A}$ and $e \in \mathcal{A}^\perp$.
- A discrete σ-subalgebra $\mathcal{F}$ is also called an *at most countable partition*.
- Consider a special case when $\mathcal{F}$ is an at most countable partition and its atoms are intervals $B_n = (b_n, b_{n-1}]$, where $1 = b_0 > b_1 > \dots$. Assume additionally that in the case of a finite partition there is an $n \in \mathbb{N}$ such that $b_n = 0$ and in the case of a countable partition $b_n \downarrow 0$. We will call such a partition an *interval partition* and denote it as $\mathcal{B} = (b_n)$.

- A countable partition B_n is called *monotonic* if $b_n - b_{n+1} > b_{n+1} - b_{n+2}, n = 0, 1, \ldots$.

- A countable partition B_n is called *monotonic* if $b_n - b_{n+1} > b_{n+1} - b_{n+2}, n = 0, 1, \ldots$.
- Two countable partitions B_n and C_n are called *equivalent* if there is a constant C, $C \geq 1$, such that

$$\frac{1}{C} \leq \frac{b_n - b_{n+1}}{c_n - c_{n+1}} \leq C, n = 0, 1, \ldots.$$

- For a σ-subalgebra $\mathcal{A}$ we define the operator of *conditional mathematical expectation*

$$E(\cdot|\mathcal{A}) : L^1 \to L^1(I, \mathcal{A}, \lambda)$$

as the Radon-Nikodym derivative

$$E(f|\mathcal{A}) = \frac{d\lambda_f}{d\lambda}, f \in L^1(I),$$

where $\lambda_f(a) = \int\limits_a f d\lambda, a \in \mathcal{A}$.

- We will also call the conditional expectation operator corresponding to a σ-subalgebra $\mathcal{A}$ the $\mathcal{A}$-*averaging operator* (omitting $\mathcal{A}$ if it will not cause any ambiguity).

- We will also call the conditional expectation operator corresponding to a σ-subalgebra $\mathcal{A}$ the $\mathcal{A}$-*averaging operator* (omitting $\mathcal{A}$ if it will not cause any ambiguity).
- After introducing some definitions we will state a norm-free version of the Calderon - Ryff real interpolation theorem.

- We will also call the conditional expectation operator corresponding to a σ-subalgebra $\mathcal{A}$ the $\mathcal{A}$-*averaging operator* (omitting $\mathcal{A}$ if it will not cause any ambiguity).
- After introducing some definitions we will state a norm-free version of the Calderon - Ryff real interpolation theorem.
- We will call a linear bounded operator $T : L^1 \to L^1$ *admissible* if $T(L^\infty) \subseteq L^\infty$.

- We will also call the conditional expectation operator corresponding to a σ-subalgebra $\mathcal{A}$ the $\mathcal{A}$-*averaging operator* (omitting $\mathcal{A}$ if it will not cause any ambiguity).
- After introducing some definitions we will state a norm-free version of the Calderon - Ryff real interpolation theorem.
- We will call a linear bounded operator $T : L^1 \to L^1$ *admissible* if $T(L^\infty) \subseteq L^\infty$.
- A positive admissible operator $U : L^1 \to L^1$ is called *double stochastic* if $U\mathbf{1} = U^\star\mathbf{1} = \mathbf{1}$ where $U^\star : L^\infty \to L^\infty$ is the Banach adjoint of U.

- We will also call the conditional expectation operator corresponding to a σ-subalgebra $\mathcal{A}$ the $\mathcal{A}$-*averaging operator* (omitting $\mathcal{A}$ if it will not cause any ambiguity).
- After introducing some definitions we will state a norm-free version of the Calderon - Ryff real interpolation theorem.
- We will call a linear bounded operator $T : L^1 \to L^1$ *admissible* if $T(L^\infty) \subseteq L^\infty$.
- A positive admissible operator $U : L^1 \to L^1$ is called *double stochastic* if $U\mathbf{1} = U^\star\mathbf{1} = \mathbf{1}$ where $U^\star : L^\infty \to L^\infty$ is the Banach adjoint of U.
- As examples of double stochastic operators we can consider composition operators generated by measure preserving endomorphisms of (I, Λ, λ) or averaging operators.

- Let X be a vector subspace of L^1 such that $L^\infty \subseteq X$. X is called an *interpolation space* if for any admissible operator T we have $TX \subseteq X$.

- Let X be a vector subspace of L^1 such that $L^\infty \subseteq X$. X is called an *interpolation space* if for any admissible operator T we have $TX \subseteq X$.

Theorem 1

(Calderon - Ryff). The following conditions are equivalent.
(1) X is an interpolation space.
(2) For any double stochastic operator U we have $UX \subseteq X$.

-
- Thus, instead of checking the property $TX \subseteq X$ for T from the enormous class of all admissible operators we can do it only for double stochastic operators. One of the main goals of this talk is to discuss how far we can go in this direction and how small can be the class of operators $\mathcal{C}$ such that if $TX \subseteq X$ for $T \in \mathcal{C}$ then X is an interpolation space.

- Let X be a vector subspace of L^1 such that $L^\infty \subseteq X$. X is called an *interpolation space* if for any admissible operator T we have $TX \subseteq X$.

Theorem 1

(Calderon - Ryff). The following conditions are equivalent.
(1) X is an interpolation space.
(2) For any double stochastic operator U we have $UX \subseteq X$.

-
- Thus, instead of checking the property $TX \subseteq X$ for T from the enormous class of all admissible operators we can do it only for double stochastic operators. One of the main goals of this talk is to discuss how far we can go in this direction and how small can be the class of operators $\mathcal{C}$ such that if $TX \subseteq X$ for $T \in \mathcal{C}$ then X is an interpolation space.
- Let us say that a σ-subalgebra $\mathcal{A}$ *averages* a vector ideal X if $E(\cdot|\mathcal{A})X \subseteq X$.

Theorem 2

Let X be an order vector ideal in L^0 such that $L^\infty \subseteq X \subseteq L^1$. The following conditions are equivalent.

1. *X is an interpolation space.*
2. *$PX \subseteq X$ for any double stochastic projector P, i.e. every σ-subalgebra of Λ averages X.*
3. *Every countable partition averages X.*

Theorem 2

Let X be an order vector ideal in L^0 such that $L^\infty \subseteq X \subseteq L^1$. The following conditions are equivalent.

1. *X is an interpolation space.*
2. *$PX \subseteq X$ for any double stochastic projector P, i.e. every σ-subalgebra of Λ averages X.*
3. *Every countable partition averages X.*

- **Remark**. It is immediate to see that every interpolation space between L^1 and L^∞ must be an order vector ideal in L^0. Therefore the Calderon - Ryff theorem follows from Theorem 2.

Theorem 2

Let X be an order vector ideal in L^0 such that $L^\infty \subseteq X \subseteq L^1$. The following conditions are equivalent.

1. *X is an interpolation space.*
2. *$PX \subseteq X$ for any double stochastic projector P, i.e. every σ-subalgebra of Λ averages X.*
3. *Every countable partition averages X.*

- **Remark**. It is immediate to see that every interpolation space between L^1 and L^∞ must be an order vector ideal in L^0. Therefore the Calderon - Ryff theorem follows from Theorem 2.
- **Remark**. It is easy to see that every finite partition averages every vector ideal between L^1 and L^∞ and therefore condition (3) in Theorem 2 is equivalent to
  (3a) Every at most countable partition averages X.

- Recall that an order vector ideal X is called *symmetric* if $f \in X \Leftrightarrow f^\star \in X$ and that a symmetric ideal X is called *principal* if there is an $f \in L^1$ such X is the smallest (by inclusion) symmetric ideal containing f.

- Recall that an order vector ideal X is called *symmetric* if $f \in X \Leftrightarrow f^\star \in X$ and that a symmetric ideal X is called *principal* if there is an $f \in L^1$ such X is the smallest (by inclusion) symmetric ideal containing f.

Definition 3

- (a) A σ-algebra $\mathcal{A}$ is called *verifying* if for any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$, the following two conditions are equivalent
  (1) $\mathcal{A}$ averages X.
  (2) X is an interpolation space.

  (b) A σ-subalgebra $\mathcal{A}$ is called *universal* if it averages any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$.

- Recall that an order vector ideal X is called *symmetric* if $f \in X \Leftrightarrow f^\star \in X$ and that a symmetric ideal X is called *principal* if there is an $f \in L^1$ such X is the smallest (by inclusion) symmetric ideal containing f.

Definition 3

- (a) A σ-algebra $\mathcal{A}$ is called *verifying* if for any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$, the following two conditions are equivalent
  (1) $\mathcal{A}$ averages X.
  (2) X is an interpolation space.

  (b) A σ-subalgebra $\mathcal{A}$ is called *universal* if it averages any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$.

- The classes $\mathfrak{U}$ of all universal σ-subalgebras and $\mathfrak{V}$ of all verifying σ-subalgebras are both nonempty and, clearly, disjoint.

- Recall that an order vector ideal X is called *symmetric* if $f \in X \Leftrightarrow f^\star \in X$ and that a symmetric ideal X is called *principal* if there is an $f \in L^1$ such X is the smallest (by inclusion) symmetric ideal containing f.

Definition 3

- (a) A σ-algebra $\mathcal{A}$ is called *verifying* if for any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$, the following two conditions are equivalent
  (1) $\mathcal{A}$ averages X.
  (2) X is an interpolation space.

  (b) A σ-subalgebra $\mathcal{A}$ is called *universal* if it averages any symmetric ideal X, $L^\infty \subseteq X \subseteq L^1$.

- The classes $\mathfrak{U}$ of all universal σ-subalgebras and $\mathfrak{V}$ of all verifying σ-subalgebras are both nonempty and, clearly, disjoint.
- The following theorem provides a complete description of *complemented* universal and *complemented* verifying σ-subalgebras of Λ.

Theorem 4

Let $\mathcal{A}$ be a complemented σ-subalgebra.
I. $\mathcal{A}$ is a universal subalgebra if and only if either $\mathcal{A}$ or its independent complement $\mathcal{A}^{\perp}$ is generated by a finite partition.
II. $\mathcal{A}$ is a verifying subalgebra if and only if one of the following conditions is satisfied.
(a) Both $\mathcal{A}$ and its complement $\mathcal{A}^{\perp}$ are not purely discrete.
(b) $\mathcal{A}$ or $\mathcal{A}^{\perp}$ is generated by a countable monotonic partition equivalent to a geometric progression.
(c)The subalgebra $\mathcal{A}$ is generated by a countable partition and there is a subalgebra $\mathcal{B}$ satisfying condition (b) and such that $\mathcal{A}$ is finer than $\mathcal{B}$.

-

Corollary 5

If $\mathcal{A}$ is generated by an at most countable partition then there exists a verifying subalgebra $\mathcal{B}$ such that $\mathcal{B}$ is finer than $\mathcal{A}$.

Theorem 4

Let $\mathcal{A}$ be a complemented σ-subalgebra.
I. $\mathcal{A}$ is a universal subalgebra if and only if either $\mathcal{A}$ or its independent complement $\mathcal{A}^\perp$ is generated by a finite partition.
II. $\mathcal{A}$ is a verifying subalgebra if and only if one of the following conditions is satisfied.
(a) Both $\mathcal{A}$ and its complement $\mathcal{A}^\perp$ are not purely discrete.
(b) $\mathcal{A}$ or $\mathcal{A}^\perp$ is generated by a countable monotonic partition equivalent to a geometric progression.
(c)The subalgebra $\mathcal{A}$ is generated by a countable partition and there is a subalgebra $\mathcal{B}$ satisfying condition (b) and such that $\mathcal{A}$ is finer than $\mathcal{B}$.

Corollary 5

If $\mathcal{A}$ is generated by an at most countable partition then there exists a verifying subalgebra $\mathcal{B}$ such that $\mathcal{B}$ is finer than $\mathcal{A}$.

Corollary 6

Let $\mathcal{A}$ be a σ-subalgebra of Λ. Then $\mathcal{A}$ is a verifying (universal) subalgebra if and only if its independent complement $\mathcal{A}^{\perp}$ is a verifying (respectively, universal) subalgebra.

-

Corollary 6

Let $\mathcal{A}$ be a σ-subalgebra of Λ. Then $\mathcal{A}$ is a verifying (universal) subalgebra if and only if its independent complement $\mathcal{A}^{\perp}$ is a verifying (respectively, universal) subalgebra.

-
- Corollary 6 is connected with the following much deeper and as yet unsolved problem.

Corollary 6

Let $\mathcal{A}$ be a σ-subalgebra of Λ. Then $\mathcal{A}$ is a verifying (universal) subalgebra if and only if its independent complement $\mathcal{A}^{\perp}$ is a verifying (respectively, universal) subalgebra.

-
- Corollary 6 is connected with the following much deeper and as yet unsolved problem.
- **Problem**. Let X be an arbitrary symmetric ideal in L^1 and $\mathcal{A}$ be an arbitrary complemented σ-subalgebra of Λ. Is it true that $\mathcal{A}$ averages X if and only if $\mathcal{A}^{\perp}$ averages X?

- The following two theorems provide some support to the case that the previous problem might have a positive solution.

- The following two theorems provide some support to the case that the previous problem might have a positive solution.

Theorem 7

Let X and Y be symmetric ideals in L^1 and $\mathcal{A}$ be a countable partition. Assume that $X \subseteq L^1 \log^+ L^1$.
Then the following implication holds.

$$E(X|\mathcal{A}) \subseteq Y \Rightarrow E(X|\mathcal{A}^{\perp}) \subseteq Y$$

-

Corollary 8

Let X be a symmetric ideal in L^1 such that $X \subseteq L^1 \log^+ L^1$. If a countable partition $\mathcal{A}$ averages X then its independent complement $\mathcal{A}^\perp$ also averages X.

-

Corollary 8

Let X be a symmetric ideal in L^1 such that $X \subseteq L^1 \log^+ L^1$. If a countable partition $\mathcal{A}$ averages X then its independent complement $\mathcal{A}^\perp$ also averages X.

-

Theorem 9

Let X be a principal symmetric ideal in L^1. If a countable partition $\mathcal{A}$ averages X then its independent complement $\mathcal{A}^\perp$ also averages X.

-

Corollary 8

Let X be a symmetric ideal in L^1 such that $X \subseteq L^1 \log^+ L^1$. If a countable partition $\mathcal{A}$ averages X then its independent complement $\mathcal{A}^\perp$ also averages X.

-

Theorem 9

Let X be a principal symmetric ideal in L^1. If a countable partition $\mathcal{A}$ averages X then its independent complement $\mathcal{A}^\perp$ also averages X.

-
- **Remark**. In addition let us mention the following quite simple fact. Let X be a symmetric ideal in L^1 and let $\mathcal{A}$ and $\mathcal{A}^\perp$ be both continuous σ-subalgebras of Λ. Then

$$E(X|\mathcal{A}) \subseteq X \Leftrightarrow E(X|\mathcal{A}^\perp) \subseteq X.$$

- Let Z be a subset of L^1. We will denote by $\mathcal{N}_Z$ the smallest symmetric ideal in L^1 that contains Z. If $f \in L^1$ then $\mathcal{N}_{\{f\}}$ is the principal symmetric ideal generated by f and we will write $\mathcal{N}_f$ instead of $\mathcal{N}_{\{f\}}$.

- Let Z be a subset of L^1. We will denote by $\mathcal{N}_Z$ the smallest symmetric ideal in L^1 that contains Z. If $f \in L^1$ then $\mathcal{N}_{\{f\}}$ is the principal symmetric ideal generated by f and we will write $\mathcal{N}_f$ instead of $\mathcal{N}_{\{f\}}$.

Definition 10

Let $f \in L^1$.
(1) f is called weakly regular if the vector order ideal in L^1 generated by f coincides with $\mathcal{N}_f$.
(2) f is called regular if $\mathcal{N}_f$ is an interpolation space.
(3) If subalgebra $\mathcal{B}$ is generated by an interval partition and $\mathcal{B}$ averages $\mathcal{N}_f$ we will call f $\mathcal{B}$-regular.

-

- **Remark**. **(1)**. If $f \in L^1(I)$ then the principal ideal $\mathcal{N}_f$ can be described as follows.

- **Remark**. **(1)**. If $f \in L^1(I)$ then the principal ideal $\mathcal{N}_f$ can be described as follows.
-

$$\mathcal{N}_f = \{g \in L^1(I) : \exists C = C(g) > 0, \text{such that a.e.} g^\star(t) \le Cf^\star(t/C)\}.$$

- **Remark**. **(1)**. If $f \in L^1(I)$ then the principal ideal $\mathcal{N}_f$ can be described as follows.
-
$$\mathcal{N}_f = \{g \in L^1(I) : \exists C = C(g) > 0, \text{such that a.e.} g^\star(t) \leq Cf^\star(t/C)\}.$$
- **(2)** A function $f \in L^1(I)$ is weakly regular if and only if there is a c, $0 < c < 1$, such that a.e.
$$f^\star(ct) \leq f^\star(t)/c.$$

- **Remark**. **(1)**. If $f \in L^1(I)$ then the principal ideal $\mathcal{N}_f$ can be described as follows.
- $$\mathcal{N}_f = \{g \in L^1(I) : \exists C = C(g) > 0, \text{such that a.e.} g^\star(t) \leq Cf^\star(t/C)\}.$$
- **(2)** A function $f \in L^1(I)$ is weakly regular if and only if there is a c, $0 < c < 1$, such that a.e.
$$f^\star(ct) \leq f^\star(t)/c.$$
- **(3)** If $f \in L^1(I)$ is weakly regular then the principal ideal $\mathcal{N}_f$ can be described as follows.
$$\mathcal{N}_f = \{g \in L^1(I) : \exists C = C(g) > 0, \text{such that a.e.} g^\star(t) \leq Cf^\star(t)\}.$$

Theorem 11

Let $f \in L^1$. The following conditions are equivalent.
(1) f is regular.
(2) If $g \in L^1$ and $f^{\star\star} \sim g^{\star\star}$ then g is weakly regular.

-

Theorem 11

Let $f \in L^1$. The following conditions are equivalent.
(1) f is regular.
(2) If $g \in L^1$ and $f^{\star\star} \sim g^{\star\star}$ then g is weakly regular.

-
- Let $\mathcal{B} = \sigma(\{B_n\})$ be the σ-subalgebra generated by the interval partition $\{B_n\}$. We will denote by $\mathcal{B}^\star$ the σ-subalgebra generated by the unique interval partition $\{B_n^\star\}$ obtained by rearranging the intervals B_n in such a way that their lengths, when going from 1 to 0, are non-increasing.

Theorem 12

Let $\mathcal{B}$ be a σ-subalgebra generated by a monotonic interval partition, i.e $\mathcal{B} = \mathcal{B}^\star$. Let $f \in L^1$. the following conditions are equivalent.
(1) f is $\mathcal{B}$-regular.
(2) The smallest symmetric ideal $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})}$ generated by $E(\mathcal{N}_f|\mathcal{B})$ is a principal symmetric ideal, i.e there is $g \in L^1$ such that $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})} = \mathcal{N}_g$.

-

Theorem 12

Let $\mathcal{B}$ be a σ-subalgebra generated by a monotonic interval partition, i.e $\mathcal{B} = \mathcal{B}^{\star}$. Let $f \in L^1$. the following conditions are equivalent.
(1) f is $\mathcal{B}$-regular.
(2) The smallest symmetric ideal $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})}$ generated by $E(\mathcal{N}_f|\mathcal{B})$ is a principal symmetric ideal, i.e there is $g \in L^1$ such that $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})} = \mathcal{N}_g$.

-
- The proof of Theorem 12 is based on the following theorem that might be of independent interest.

Theorem 12

Let $\mathcal{B}$ be a σ-subalgebra generated by a monotonic interval partition, i.e $\mathcal{B} = \mathcal{B}^{\star}$. Let $f \in L^1$. the following conditions are equivalent.
(1) f is $\mathcal{B}$-regular.
(2) The smallest symmetric ideal $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})}$ generated by $E(\mathcal{N}_f|\mathcal{B})$ is a principal symmetric ideal, i.e there is $g \in L^1$ such that $\mathcal{N}_{E(\mathcal{N}_f|\mathcal{B})} = \mathcal{N}_g$.

-
- The proof of Theorem 12 is based on the following theorem that might be of independent interest.

Theorem 13

Every interval partition $\{B_n\}$ is equivalent to a subsequence of $\{B_n^{\star}\}$. Moreover, the smallest constant of equivalence is the golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2}.$$