Order Continuous Operators on pre-Riesz Spaces

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Content:

pre-Riesz spaces

How to generalize structures from vector lattices to ordered vector spaces?

Definition (Buskes – van Rooij)

Let Z be an ordered vector space and $X \subseteq Z$ a linear subspace.
X is **order dense** in Z, if for every $z \in Z$ we have $z = \inf \{x \in X \mid z \leq x\}$.

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Definition (Theorem by van Haandel, 1993)

An ordered vector space X is a **pre-Riesz space** if there exists a vector lattice Z and a bipositive linear mapping $i : X \to Z$ (i.e. i is an embedding) such that $i(X)$ order dense in Z.
(Z, i) is called a **vector lattice cover** of X.

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Examples:

1. $C^1[0,1]$ is order dense in $C[0,1]$,
2. ℓ_0^∞ (vector space of eventually constant sequences) is order dense in ℓ^∞.

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1. $C^1[0, 1]$ is order dense in $C[0, 1]$,
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Let X be an Archimedean vector lattice, $x, y \in X$ and $S \subseteq X$.
$S^u := \{x \in X \mid x \geq S\}$ – set of all upper bounds of S.

Definition

x and y are **disjoint** (in symbols $x \perp y$) if $|x| \wedge |y| = 0$ (iff $|x + y| = |x - y|$).

A subset $B \subseteq X$ is a **band**, if $B = B^{\mathrm{dd}}$.

Let X be a pre-Riesz space with a vector lattice cover (Z, i), $x, y \in X$ and $S \subseteq X$.
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Theorem (Kalauch – van Gaans, 2006)

$x \perp y \quad \Leftrightarrow \quad i(x) \perp i(y).$

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Theorem (van Haandel, 1993)

Let X be an ordered vector space.

- *If X is directed and Archimedean, then X is pre-Riesz.*
- *If X is pre-Riesz, then X is directed.*

From here on: only *Archimedean* pre-Riesz spaces and vector lattices.

Content:

Let X and Y be pre-Riesz spaces, $(x_\alpha)_\alpha$ a net in X and $T\colon X \to Y$ a linear operator.

Definition

T is **regular** if there exist positive operators $T_1, T_2\colon X \to Y$ with $L_r(X,Y)$

$$T = T_1 - T_2,$$

T is **order continuous** if $x_\alpha \xrightarrow{o} x$ implies $T(x_\alpha) \xrightarrow{o} T(x)$. $L_{oc}(X,Y)$

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$$\left.\begin{array}{lll} X \text{ is directed} & \Rightarrow & L_r(X,Y) \text{ is directed} \\ Y \text{ is Archimedean} & \Rightarrow & L_r(X,Y) \text{ is Archimedean} \end{array}\right\} \Rightarrow L_r(X,Y) \text{ is pre-Riesz.}$$

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Task: Find vector lattice covers of $L_r(X,Y)$ and $L_{oc}^{\diamond}(X,Y)$ which **consist of operators**.

Recall:

Theorem (Riesz – Kantorovich)

Let Z_1 be a directed ordered vector space with the Riesz Decomposition Property and Z_2 be a Dedekind complete vector lattice.
Then $L_r(Z_1, Z_2)$ is a Dedekind complete vector lattice.

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Idea: Make the range space Dedekind complete!

Let X and Y be pre-Riesz spaces and let X have the RDP. Then

$$\underset{\text{pre-Riesz space}}{L_r(X, Y)} \subseteq \underset{\text{vector lattice}}{L_r(X, Y^\delta)}$$

order dense?

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order dense?			**order dense?**	

No, not even under strong additional conditions!

$(X := \ell_0^\infty, Y := \ell_0^\infty)$

Proposition (Abramovich – Wickstead, 1991)

The ordered vector space $L_r(\ell_0^\infty)$ does not have the RDP and therefore is not a vector lattice.

Proposition

The ordered vector space $L_{oc}^\diamond(\ell_0^\infty)$ is not a vector lattice.

Example

$L_r(\ell_0^\infty)$ is not majorizing and thus not order dense in $L_r(\ell_0^\infty, \ell^\infty)$.

Let $T\colon \ell_0^\infty \to \ell^\infty$ be defined by

$b \in B$	$T(b)$														
e_1	(**1**	0	**1**	0	0	**1**	0	0	0	**1**	0	0	0	0	$\dots$)
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$$\Rightarrow \qquad n = \sum_{i=1}^{n} \limsup T(e_i) \leq \sum_{i=1}^{n} \lim S(e_i) = \lim \left(\sum_{i=1}^{n} S(e_i) \right)$$

$\sum_{i=1}^n e_i \leq \mathbb{1}$ implies $\sum_{i=1}^n S(e_i) \leq S(\mathbb{1})$

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$$\Rightarrow \qquad n = \sum_{i=1}^{n} \limsup T(e_i) \le \sum_{i=1}^{n} \lim S(e_i) = \lim \left(\sum_{i=1}^{n} S(e_i) \right) \le \lim S(\mathbb{1}). \ \lightning$$

$\sum_{i=1}^n e_i \le \mathbb{1}$ implies $\sum_{i=1}^n S(e_i) \le S(\mathbb{1})$

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$L_r(\ell_0^\infty)$ is not majorizing and thus not order dense in $L_r(\ell_0^\infty, \ell^\infty)$.

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Show: The operator T in the previous example is order continuous.

ℓ_0^∞ has nice properties:

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Content:

Let X and Y be pre-Riesz spaces and let X have the RDP.

Better idea: Make $L_r(X,Y)$ and $L_{oc}^{\diamond}(X,Y)$ majorizing!

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Let X be a pre-Riesz space and (Z, i) a vector lattice cover of X.

Definition

An element $a \in X_+ \setminus \{0\}$ is called an **atom** if

$$\forall\, x \in X\colon 0 < x \leq a \quad \Rightarrow \quad \exists \lambda \in \mathbb{R}_{>0}\colon x = \lambda a.$$

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Theorem (Dedekind completion of $L_{oc}^{\diamond}$)

Let X and Y be pre-Riesz spaces and let X be atomic, pervasive and have the RDP. Then $L_{oc}^{\diamond}(X,Y)$ has a vector lattice cover consisting of operators, namely the ideal

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Example: For any pre-Riesz space Y the Dedekind completion of $L_{oc}^{\diamond}(\ell_0^{\infty},Y)$ is the ideal

$$\mathcal{I}_{L_{oc}^{\diamond}(\ell_0^{\infty},Y)} = \left\{T \in L_{oc}(\ell_0^{\infty},Y^{\delta}) \;\middle|\; \exists S \in L_{oc}^{\diamond}(\ell_0^{\infty},Y) \colon \; |T| \leq S\right\}.$$

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Sketch of the proof:

- Show: sufficient to approximate every $T \in J_+$ from above, i.e. to show $T = \inf\{S \in L_{oc}^{\diamond} \mid T \leq S\}$.
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$$S_y^{(a)}(x) := \begin{cases} S(x) & \text{for } x \in \{0\} \oplus \mathcal{B}_a^{\mathrm{d}} \\ \lambda y & \text{for } x = \lambda a,\ x \in \mathcal{B}_a \oplus \{0\}\,. \end{cases}$$

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In particular, J is the Dedekind completion of $L_{oc}^{\diamond}(X,Y)$.

Theorem

Let X and Y as above.

If for $U,V \in L_{oc}^{\diamond}(X,Y)$ the supremum $U \vee V$ or the infimum $U \wedge V$ exists in $L_{oc}^{\diamond}(X,Y)$, then it can be computed by the Riesz-Kantorovich formulae.

Proposition

Let X be a atomic vector lattice with an algebraic basis consisting of atoms, i.e. $X = \operatorname{lin}\{a \in X \mid a \text{ is an atom}\}$. Let Y be pre-Riesz. Then $L_r(X,Y) = L_{oc}^{\diamond}(X,Y)$.

Corollary (Dedekind completion of L_r)

Let X and Y be as above. Then $L_r(X,Y)$ has a vector lattice cover consisting of operators, namely the ideal

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Thank you for your attention!