Vector-valued extrapolation in Banach function spaces

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Joint work with Alex Amenta and Mark Veraar

Delft University of Technology

A. Amenta, E. Lorist, and M. C. Veraar. “Rescaled extrapolation for vector-valued functions”. arXiv:1703.06044. 2017

A. Amenta, E. Lorist, and M. C. Veraar. “Fourier multipliers in Banach function spaces with UMD concavifications”. arXiv:1705.07792. 2017

Overview

Bochner spaces

Definition

Let X be a Banach space. Let $p \in [1, \infty)$, then the *Bochner space* $L^p(\mathbb{R}^d; X)$ consists of all strongly measurable functions $f : \mathbb{R}^d \to X$ such that

$$\|f\|_{L^p(\mathbb{R}^d;X)} := \left(\int_{\mathbb{R}^d} \|f(x)\|_X^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Example

If X is a Banach function space over measure space (S, μ), then $L^p(\mathbb{R}^d; X)$ consists of $f : \mathbb{R}^d \times S \to \mathbb{C}$ such that

$$f(x, \bullet) \in X, \qquad \text{a.e. } x \in \mathbb{R}^d$$

$$x \mapsto \|f(x, \bullet)\|_X \in L^p(\mathbb{R}^d)$$

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Extension problem

Let $p \in [1, \infty)$ and X be a Banach space. For a T bounded linear operator on $L^p(\mathbb{R}^d)$ we define $\widetilde{T} = T \otimes I_X$ on $L^p(\mathbb{R}^d) \otimes X$ by

$$\widetilde{T}(f \otimes e) = Tf \otimes e$$

Main Question

When does $\widetilde{T}$ define a bounded operator on $L^p(\mathbb{R}^d; X)$?

- If $X = L^p$ or X is a Hilbert space.
- If T is positive.
- In general not even for $p = 2$.

Goal

Provide sufficient conditions on X and T.

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Fourier Multipliers

We let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$. For a function $m : \mathbb{R}^d \to \mathbb{C}$ define the following operator

$$T_m f := \mathcal{F}^{-1}(m\mathcal{F}(f))$$

Examples

The Hilbert transform "$\frac{\mathrm{d}}{\mathrm{d}x} / |\frac{\mathrm{d}}{\mathrm{d}x}|$" for $d = 1$:

$$H := T_m \qquad \text{with} \qquad m(\xi) = -i\,\mathrm{sgn}(\xi)$$

The Riesz transforms:

$$R_j := T_m \qquad \text{with} \qquad m(\xi) = -i\frac{\xi_j}{|\xi|}$$

Classifying which m yield bounded operators on L^p is **very** delicate!

Theorem (M. Riesz '28)

H is bounded on $L^p(\mathbb{R})$ and R_j is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

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The UMD property

Is $\widetilde{H}$ bounded on $L^p(\mathbb{R}; X)$?

Definition (Burkholder, Bourgain '83)

A Banach space X is said to have the UMD property if $\widetilde{H}$ is bounded on $L^p(\mathbb{R}; X)$ for some (all) $p \in (1, \infty)$.

Equivalent to **U**nconditionality of **M**artingale **D**ifferences

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The following spaces have the UMD property:

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Muckenhoupt weights

Definition

$w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is called a weight if $w > 0$. Let $p \in (1,\infty)$, then $L^p(\mathbb{R}^d, w)$ consist of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^d,w)} := \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)\,\mathrm{d}x\right)^{\frac{1}{p}} < \infty$$

We consider the classes of Muckenhoupt weights A_p for $p \in (1,\infty)$.

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Theorem

Let w be a weight. The following are equivalent

- $w \in A_p$.
- The Hilbert transform is bounded on $L^p(\mathbb{R}, w)$ (for $d = 1$).
- The Riesz projections are bounded on $L^p(\mathbb{R}^d, w)$.
- The Hardy-Littlewood maximal function is bounded on $L^p(\mathbb{R}^d, w)$.

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Example

Let $w(x) = |x|^\alpha$ for $x \in \mathbb{R}^d$, then

$$w \in A_p \quad \Longleftrightarrow \quad -d < \alpha < d(p-1)$$

Scalar-valued Extrapolation

Theorem (Rubio de Francia '84)

Let T be a sublinear operator on $L^p(\mathbb{R}^d, w)$.

T is bounded on $L^p(\mathbb{R}^d, w)$ for **some** $p \in (1, \infty)$ and **all** $w \in A_p$.

$$\Longrightarrow$$

T is bounded on $L^p(\mathbb{R}^d, w)$ for **all** $p \in (1, \infty)$ and **all** $w \in A_p$.

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p-Convexity

Definition

Let $p \in [1, \infty]$. A Banach function space X is p-convex if

$$\Big\| \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p} \Big\|_X \leq \Big(\sum_{k=1}^{n} \|x_k\|_X^p \Big)^{1/p}$$

for all $x_1, \cdots, x_n \in X$. It is p-concave if the reverse estimate holds.

- Every Banach function space is 1-convex and ∞-concave.
- If a Banach function space is p-convex and q-concave, then
  - $p \leq q$.
  - X is p_0-convex for all $p_0 \leq p$.
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Concavification

Definition

Let $p \in (0, \infty)$ and let X be a p-convex Banach function space. Define the p-concavification X^p of X by

$$X^p = \left\{ |x|^p \operatorname{sgn}(x) : x \in X \right\} = \left\{ x : |x|^{1/p} \in X \right\}$$

with the norm $\|x\|_{X^p} = \| |x|^{1/p} \|_X^p$

For example $(L^p)^r = L^{p/r}$.

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Vector-valued extrapolation

Theorem (Rubio de Francia '86)

Let T be a sublinear operator on $L^p(\mathbb{R}, w)$ and let X be a Banach function space with the UMD property.

T is bounded on $L^p(\mathbb{R}, w)$ for **all** $p \in (1, \infty)$ and **all** $w \in A_p$

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Here $\widetilde{T}$ is defined on simple functions $f : \mathbb{R} \to X$ by

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and extended by density. For linear operators this coincides with tensor extension $\widetilde{T} = T \otimes I_X$.

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Theorem (Amenta, L., Veraar '17)

Let T be a sublinear operator on $L^p(\mathbb{R}^d, w)$. Take $p_0 \in (0, \infty)$ and let X be a Banach function space such that X^{p_0} has the UMD property.

T is bounded on $L^p(\mathbb{R}^d, w)$ for **all** $p \in (p_0, \infty)$ and **all** $w \in A_{p/p_0}$.

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Applications and conclusion

- Vector-valued Littlewood–Paley–Rubio de Francia inequalities.
  (Rubio de Francia 85'), (Potapov, Sukochev, Xu '12), (Król '14)
- Operator-valued Fourier multiplier theorems.
  (Hytönen, Potapov '06), (Król '14)
- Boundedness of the vector-valued variational Carleson operator.
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Provide sufficient conditions on X and T such that $\widetilde{T}$ is a bounded operator on $L^p(\mathbb{R}^d;X)$

- X a Banach function space such that X^{p_0} is a Banach function space with the UMD property
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Thank you for your attention!

LPR for Banach function spaces

For an interval $I \subset \mathbb{R}$ we define

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Theorem (Rubio de Francia '85)

Let $p \in (2, \infty)$ and let $\mathcal{I}$ be a collection of mutually disjoint intervals in $\mathbb{R}$, then for $w \in A_{p/2}$ and $f \in L^p(w)$

$$\Big\| \Big(\sum_{I \in \mathcal{I}} |S_I f|^2 \Big)^{1/2} \Big\|_{L^p(w)} \lesssim_{p,w} \|f\|_{L^p(w)}$$

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Theorem (Rubio de Francia '85, Król '14)

Let $q \in [2, \infty)$, $p \in (q', \infty)$ and let $\mathcal{I}$ be a collection of mutually disjoint intervals in $\mathbb{R}$, then for $w \in A_{p/q'}$ and $f \in L^p(w)$

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Let X be a Banach function space and $p \in (q', \infty)$. X has the $\mathrm{LPR}_{p,q}$ property if

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Theorem (Potapov, Sukochev, Xu '12)

Let X be a Banach function space such that X^2 is a Banach function space with the UMD property. Then X has the $\mathrm{LPR}_{p,2}$ property for all $p \in (2, \infty)$.

Corollary (Amenta, L., Veraar '17)

Let $q \in [2, \infty)$ and let X be a Banach function space such that $X^{q'}$ is a Banach function space with the UMD property. Then X has the $\mathrm{LPR}_{p,q}$ property for all $p \in (q', \infty)$. Moreover, the defining estimate holds for all $w \in A_{p/q'}$ and $f \in L^p(w; X)$.

As an application we obtain an operator-valued Fourier multiplier result, which extends a result of (Hytönen, Potapov '06).

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Scalar-valued limited range extrapolation

Theorem (Auscher, Martell '07)

Let T be a sublinear operator on $L^p(w)$ and $0 \leq p_- < p_+ \leq \infty$.

T is bounded on $L^p(w)$ for **some** $p \in (p_-, p_+)$ and **all** $w \in A_{p/p_-} \cap RH_{(p_+/p)'}$.

$$\Longrightarrow$$

T is bounded on $L^p(w)$ for **all** $p \in (p_-, p_+)$ and **all** $w \in A_{p/p_-} \cap RH_{(p_+/p)'}$.

Vector-valued limited range extrapolation

Theorem (L. '17)

Let T be a sublinear operator on $L^p(w)$. Take $0 \leq p_- < p_+ \leq \infty$ and let X be p_--convex, p_+-concave Banach function space such that $\big((X^{p_-})^*\big)^{p_+/p_-}$ has the UMD property.

T is bounded on $L^p(w)$ for **all** $p \in (p_-, p_+)$ and **all** $w \in A_{p/p_-} \cap RH_{(p_+/p)'}$.

$$\Longrightarrow$$

$\widetilde{T}$ is bounded on $L^p(w;X)$ for **all** $p \in (p_-, p_+)$ and **all** $w \in A_{p/p_-} \cap RH_{(p_+/p)'}$.

Let $X = L^q$, then $\big((X^{p_-})^*\big)^{p_+/p_-}$ has the UMD property if and only if

$$p_- < q < p_+$$