Some loose ends on unbounded order convergence

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1 Motivation

2 Main results

Unbounded order convergence

- In 1948, a type of order convergence was introduced by Nakano in semi-ordered linear spaces, in order to establish a version of Birkhoff's Ergodic Theorem in the setting of partially ordered spaces: Analogue of a.e. convergence
- In 1977, Wickstead introduced it into Banach lattices and named it unbounded order convergence.

Definition

Let X be a vector lattice, a net (x_α) in X is said to **unbounded order converge** to $x \in X$, $x_\alpha \xrightarrow{\text{uo}} x$, if $|x_\alpha - x| \wedge y \xrightarrow{\text{o}} 0$ for any $y \in X_+$.

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Gao and Xanthos (2014) used it to study Doob's Martingale Convergence Theorem in a general framework of vector and Banach lattices;

Theorem(Dobb)

Every norm bounded submartingale in $L_1(\mu)$ converges almost surely (to a limit in $L_1(\mu)$).

Let X be a vector lattice, a filtration (E_n) on X is a sequence positive projections on X such that $E_nE_m = E_mE_n = E_{m\wedge n}$ for all $m,\ n \geq 1$. Recall also that a sequence $(x_n) \subset X$ is called a martingale relative to (E_n) if $E_nx_m = x_n$ for all $m \geq n$.

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Theorem(Gao and Xanthos, 2014)

Let X be a vector lattice with a weak unit and a strictly positive order continuous functional, then every martingale (z_n) in $L_1(\Omega, X)$ with respect to a classical filtration is almost surely uo-Cauchy in X.

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a net $\{x_\alpha\}$ is said to be unbounded order Cauchy (or, Uo-Cauchy for short), if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')}$ uo-convergences to 0.

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Let X be an order continuous Banach lattice. Then every norm bounded uo-Cauchy net is uo-convergent $\Longleftrightarrow$ X is KB, every norm bounded increasing net is convergent (in order and in norms).

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In vector lattice X, is a norm bounded increasing net uo-Cauchy ?

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Can we find a limit for a uo-Cauchy net? Say, in the universal completion?

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Uo-Cauchy net

Theorem 1

Let X be a vector lattice.Suppose that $X_n^\sim$ separates points of X, then every norm bounded positive increasing net (x_α) is uo-Cauchy in X.

Step 1: WLOG, assume X is order complete.

- Let $\{\phi_\gamma\}$ be a maximal disjoint collection in $(X_n^\sim)_+$;
- For each γ, the null idea of ϕ_γ is $N_\gamma = \{x \in X : \phi_\gamma(|x|) = 0\}$; the carrier is $C_\gamma = N_\gamma^d$;
- $X \backsim \oplus C_\gamma$; pass to C_γ by considering $(P_\gamma x_\alpha)_\alpha$.

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The following lemma guarantees we can just need to prove each $(P_\gamma x_\alpha)_\alpha$ is uo-Cauchy in C_γ: simply let $D = \bigcup_\gamma C_\gamma$ and notice $|x_\alpha - x_{\alpha'}| \wedge y = |P_\gamma x_\alpha - P_\gamma x_{\alpha'}| \wedge y$ for each $y \in C_\gamma$.

Lemma

Let X be a vector lattice and D be a set in X_+. TFAE:

1. The band generated by D is X.
2. For any net (x_α) in X_+, $x_\alpha \wedge d \xrightarrow{o} 0$ for any $d \in D$ implies $x_\alpha \xrightarrow{uo} 0$.

- On C_γ, ϕ_γ is a strictly positive order continuous functional. Then C_γ can be embedded in the $L_1(\mu)$ space-the norm completion of $(C_\gamma, ||\cdot||_\gamma)$ in which $||y||_\gamma = \phi_\gamma(|y|)$ for each $y \in C_\gamma$.

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corollary

Let X be a Banach function space over a σ-finite measure space. Then any norm bounded positive increasing sequence in X converges a.e. to a real-valued measurable function.

Theorem 2

Let X be a vector lattice such that $X_n^\sim$ separates points of X. Then X^{u} is uo-complete, and every uo-Cauchy net in X is uo-convergent in X^{u}.

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Universal completion

Recall that a vector lattice X is said to be:

- laterally complete if every collection of mutually disjoint positive vectors admit a supremum;
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- $X \backsim \oplus C_\gamma$;
- Moreover, $X \backsim \oplus B_\sigma$ and for each σ, B_σ is a principal band;

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Countable sup property

Step 2:

Lemma

Let X be a vector lattice with a weak unit $u > 0$. If X has the countable sup property, then X^{u} also has the countable sup property.

Step 3:

Proposition

Let X be an order complete vector lattice. If X is universally complete, and, in addition, has the countable sup property, then it is uo-complete.

Remark:

- If X is uo-complete, then it is universally complete.

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Theorem 3

Let X be a vector lattice, and D be a maximal collection of disjoint positive nonzero vectors in X. Suppose that the band B_d generated by d has the countable sup property for each $d \in D$. Then X^{u} is uo-complete, and every uo-Cauchy net in X is uo-convergent in X^{u}.

Thanks for your attention!