Some loose ends on unbounded order convergence

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1 Motivation

2 Main results
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Unbounded order convergence

- In 1948, a type of order convergence was introduced by Nakano in semi-ordered linear spaces, in order to establish a version of Birkhoff’s Ergodic Theorem in the setting of partially ordered spaces: Analogue of a.e. convergence

- In 1977, Wickstead introduced it into Banach lattices and named it unbounded order convergence.

**Definition**

Let $X$ be a vector lattice, a net $(x_\alpha)$ in $X$ is said to unbounded order converge to $x \in X$, $x_\alpha \uoc x$, if $|x_\alpha - x| \wedge y \to 0$ for any $y \in X_+$. 
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Gao and Xanthos (2014) used it to study Doob’s Martingale Convergence Theorem in a general framework of vector and Banach lattices;

**Theorem (Dobb)**

Every norm bounded submartingale in $L_1(\mu)$ converges almost surely (to a limit in $L_1(\mu)$).

Let $X$ be a vector lattice, a filtration $(E_n)$ on $X$ is a sequence positive projections on $X$ such that $E_nE_m = E_mE_n = E_{m\wedge n}$ for all $m, n \geq 1$. Recall also that a sequence $(x_n) \subset X$ is called a martingale relative to $(E_n)$ if $E_nx_m = x_n$ for all $m \geq n$. 
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**Theorem (Gao and Xanthos, 2014)**

Let $X$ be a vector lattice with a weak unit and a strictly positive order continuous functional, then every martingale $(z_n)$ in $L_1(\Omega, X)$ with respect to a classical filtration is almost surely uo-Cauchy in $X$.

**Uo-Cauchy**

A net $\{x_\alpha\}$ is said to be unbounded order Cauchy (or, Uo-Cauchy for short), if the net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha')}$ uo-converges to 0.
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Let $X$ be an order continuous Banach lattice. Then every norm bounded uo-Cauchy net is uo-convergent $\iff X$ is KB, every norm bounded increasing net is convergent (in order and in norms).

**Question 1**

In vector lattice $X$, is a norm bounded increasing net uo-Cauchy?

**Question 2**

Can we find a limit for a uo-Cauchy net? Say, in the universal completion?
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Let $X$ be a vector lattice. Suppose that $X_n^\sim$ separates points of $X$, then every norm bounded positive increasing net $(x_\alpha)$ is uo-Cauchy in $X$.

Step 1: WLOG, assume $X$ is order complete.

- Let $\{\phi_\gamma\}$ be a maximal disjoint collection in $(X_n^\sim)_+$;
- For each $\gamma$, the null idea of $\phi_\gamma$ is $\mathcal{N}_\gamma = \{x \in X : \phi_\gamma(|x|) = 0\}$; the carrier is $C_\gamma = \mathcal{N}_\gamma^d$;
- $X \sim \oplus C_\gamma$; pass to $C_\gamma$ by considering $(P_\gamma x_\alpha)_\alpha$. 
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- $X \sim \bigoplus C_\gamma$; pass to $C_\gamma$ by considering $(P_\gamma x_\alpha)_\alpha$. 
The following lemma guarantees we can just need to prove each \((P_\gamma x_\alpha)_\alpha\) is \(uo\)-Cauchy in \(C_\gamma\): simply let \(D = \bigcup_\gamma C_\gamma\) and notice \(|x_\alpha - x_\alpha'| \wedge y = |P_\gamma x_\alpha - P_\gamma x_\alpha'| \wedge y\) for each \(y \in C_\gamma\).

**Lemma**

Let \(X\) be a vector lattice and \(D\) be a set in \(X_+\). TFAE:

1. The band generated by \(D\) is \(X\).
2. For any net \((x_\alpha)\) in \(X_+\), \(x_\alpha \wedge d \xrightarrow{0} 0\) for any \(d \in D\) implies \(x_\alpha \xrightarrow{uo} 0\).
On $C_\gamma$, $\phi_\gamma$ is a strictly positive order continuous functional. Then $C_\gamma$ can be embedded in the $L_1(\mu)$ space—the norm completion of $(C_\gamma, \| \cdot \|_\gamma)$ in which $\| y \|_\gamma = \phi_\gamma(\| y \|)$ for each $y \in C_\gamma$. 
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corollary

Let $X$ be a Banach function space over a $\sigma$-finite measure space. Then any norm bounded positive increasing sequence in $X$ converges a.e. to a real-valued measurable function.

Theorem 2

Let $X$ be a vector lattice such that $X_\sim^n$ separates points of $X$. Then $X^u$ is uo-complete, and every uo-Cauchy net in $X$ is uo-convergent in $X^u$. 
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Universal completion

Recall that a vector lattice $X$ is said to be:

- laterally complete if every collection of mutually disjoint positive vectors admit a supremum;
- universally complete if it is order complete and laterally complete.
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- $X \sim \bigoplus C_{\gamma}$;
- Moreover, $X \sim \bigoplus B_{\sigma}$ and for each $\sigma$, $B_{\sigma}$ is a principal band;
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**Lemma**

Let $X$ be a vector lattice with a weak unit $u > 0$. If $X$ has the countable sup property, then $X^u$ also has the countable sup property.
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**Proposition**

Let $X$ be an order complete vector lattice. If $X$ is universally complete, and, in addition, has the countable sup property, then it is uo-complete.

**Remark:**

- If $X$ is uo-complete, then it is universally complete.
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Theorem 3

Let $X$ be a vector lattice, and $D$ be a maximal collection of disjoint positive nonzero vectors in $X$. Suppose that the band $B_d$ generated by $d$ has the countable sup property for each $d \in D$. Then $X^u$ is uo-complete, and every uo-Cauchy net in $X$ is uo-convergent in $X^u$. 
Thanks for your attention!