Order and Topology of Convex Sets with Applications to Risk Measures

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Positivity IX
University of Alberta
July 2017

Joint work with Niushan Gao, Cosimo Munari and Foivos Xanthos

Risk measures

$(\Omega, \Sigma, \mathbb{P})$ atomless probability space.

Space of financial assets: X, subspace of $L^0(\Omega, \Sigma, \mathbb{P})$ containing 1 so that $|f| \leq |g|$, $g \in X \implies f \in X$.

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A *coherent risk measure* is a functional $\rho : X \to (-\infty, \infty]$ such that

1. $\rho(f + m) = \rho(f) - m$ for all $f \in X$ and all $m \in \mathbb{R}$.
2. $f \geq g,\ f, g \in X \implies \rho(f) \leq \rho(g)$.
3. $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in X$,
4. $\rho(\lambda f) = \lambda \rho(f)$ for all $f \in X$ and all $0 \leq \lambda \in \mathbb{R}$.

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4. $\rho(\lambda f) = \lambda \rho(f)$ for all $f \in X$ and all $0 \leq \lambda \in \mathbb{R}$.

A coherent risk measure is completely determined by the convex cone $C = \{f \in X : \rho(f) \leq 0\}$.

Fenchel-Moreau duality

(Fenchel-Moreau duality) Let (X, τ) be a LCTVS and let $\rho : X \to (-\infty, \infty]$ be convex and proper (not identically ∞). Define ρ^* on X^* by

$$\rho^*(\varphi) = \sup\{\varphi(f) - \rho(f) : f \in X\}.$$

Then

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The convergence described is called *order convergence*, $f_n \xrightarrow{o} f$.

Main problem

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Translated into language of convex sets: ((X, τ) has P)

(X, τ) LCTVS, order ideal of $\subseteq L^0(\Omega, \Sigma, \mathbb{P})$ containing constants: $1 \in X$, $|f| \leq |g|$, $g \in X \implies f \in X$.

C convex set in X, order closed ($=$ closed under dominated convergence) $\implies$ C is τ-closed.

Main Problem: Which (X, τ) has P?

Known examples

(X, τ) has P if

(X, τ) LCTVS $\subseteq L^0(\Omega, \Sigma, \mathbb{P})$, $1 \in X$, $|f| \leq |g|$, $g \in X \implies f \in X$.
C convex set in X, order closed
(i.e., $f_n \in C$, $f_n \xrightarrow{o} f \in X \implies f \in C$) $\implies$ C is τ-closed.

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1. X Banach lattice under natural order in $L^0(\Omega, \Sigma, \mathbb{P})$. Then $(X, \|\cdot\|)$ and (X, weak) have P.

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1. X Banach lattice under natural order in $L^0(\Omega,\Sigma,\mathbb{P})$. Then $(X,\|\cdot\|)$ and (X,weak) have P.
2. (Delbaen) $(L^\infty(\Omega,\Sigma,\mathbb{P}),\sigma(L^\infty,L^1))$ has P.

Orlicz spaces

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Let $\Phi : [0,\infty) \to \mathbb{R}$ be increasing, convex with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. The *Orlicz space*

$$L^\Phi = \{f : \exists \lambda < \infty \text{ s.t. } \int \Phi(\frac{|f|}{\lambda})\, d\mathbb{P} \leq 1\}.$$

The smallest λ in the inequality above is $\|f\|_\Phi$.
The subspace consisting of all $f \in L^\Phi$ such that the integral above is finite for all $\lambda > 0$ is called the *Orlicz heart* H^Φ.
It is the norm closure of L^∞ in L^Φ.

Φ satisfies the Δ_2*-condition* ($\Phi \in \Delta_2$) if $\limsup_{t\to\infty} \frac{\Phi(2t)}{\Phi(t)} < \infty$.

Duality of Orlicz spaces

Let Φ be an Orlicz function such that $L^{\Phi} \neq L^{1}$. The *conjugate Orlicz function* Ψ is given by

$$\Psi(t) = \sup\{ts - \Phi(s) : 0 \leq s < \infty\}.$$

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Facts:

1. $\Phi \in \Delta_2 \iff L^\Phi = H^\Phi \iff L^\Phi$ does not contain a lattice isomorphic copy of ℓ^∞.
2. $(H^\Psi)^* = L^\Phi$.
3. $L^\Psi \subseteq (L^\Phi)^*$, with equality if and only if $\Phi \in \Delta_2$.
4. $\Psi \in \Delta_2$ if and only if L^Φ does not contain a lattice isomorphic copy of ℓ^1.

Which $(L^{\Phi}, \sigma(L^{\Phi}, H^{\Psi}))$ has P?

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(Subsequence splitting principle) Suppose that $f_n \in L^\Phi$ is norm bounded and $f_n \to 0$ a.e. Then there is a subsequence (f_{n_k}) and a decomposition $f_{n_k} = g_k + h_k$, where (g_k) is pairwise disjoint, (h_k) is order bounded in L^Φ and $g_k h_k = 0$ for all k.

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If $\Psi \in \Delta_2$, then $\ell^1 \not\lesssim L^\Phi$, then $g_k \to 0$ weakly and hence some convex average (u_k) of (f_{n_k}) is order bounded and $u_k \to 0$ a.e. So $u_k \xrightarrow{o} 0$.

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Conclusion: Assume that $\Psi \in \Delta_2$.
1. Let C be a norm bounded convex set in L^Φ. If f lies in the $\sigma(L^\Phi, H^\Psi)$-closure of C, then there is a sequence (f_n) in C, dominated in L^Φ, that converges to f a.e.

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Remark: In fact, it is enough to use Cesaro averages of f_{n_k} if we use the fact that $\Phi \in \Delta_2 \implies L^\Phi$ has an upper p-estimate.

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Use Krein-Smulyan!

If $\Psi \in \Delta_2$, every order closed convex set is $\sigma(L^\Phi, H^\Psi)$-closed.
$(L^\Phi, \sigma(L^\Phi, H^\Psi))$ has P.
(Independently proved by Delbaen & Owari.)

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Converse also holds.

Theorem

TFAE.

1. Every order closed convex set in L^Φ is $\sigma(L^\Phi, H^\Psi)$-closed.
2. $\Psi \in \Delta_2$.

Which $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has P?

Easy:

1. If $\Phi \in \Delta_2$, then $L^\Phi = H^\Phi$ and so $\sigma(L^\Phi, L^\Psi)$ is the weak topology. It has P.

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Biagini & Fritelli (2009) claimed that $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ always has P. But a gap in the proof was soon noticed.

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Gao and Xanthos (preprint 2015, to appear) produced a large class of Φ so that $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ fails P.

Which $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has P?

Start with a *norm bounded* order closed convex $C \subseteq L^\Phi$. Assume that $0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)}$, can we show $0 \in C$?

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Subsequence splitting doesn't work.

If $\Psi \notin \Delta_2$, then $\ell^1 \leq L^\Phi$. Let (f_n) be a disjoint ℓ^1 sequence in L^Φ. Then $f_n \to 0$ a.e. but no average of (f_n) can be order bounded.

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1. Choose $f \in C$ so that $\int |f| g \leq 1$.
2. Split f as $f = f\chi_{\{|f| > m\}} + f\chi_{\{|f| \leq m\}} = f_1^{g,n} + f_2^{g,n}$,

where m is chosen so large that $\int_{\{|f|>m\}} \Phi(|f|) \leq \frac{1}{2^n}$.

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Take convex average. Get a_n, b_n so that $a_n + b_n \in C$ and that $\int \Phi(|a_n|) \leq \frac{1}{2^n}$, $\|b_n\|_\Phi \leq \frac{1}{2^n}$.

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Then (pointwise sum) $\sum |a_n| + \sum |b_n| \in L^\Phi$.

Also, $a_n + b_n \to 0$ in measure $\implies$ subsequence $\to 0$ a.e. (and order bounded)
Thus $0 \in C$ since C is order closed.

Krein-Smulyan property

Recall: $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has P $\overset{\text{def}}{\iff}$ every order closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.

Krein-Smulyan property

Recall: $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has $P \overset{\text{def}}{\Longleftrightarrow}$ every order closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.

We have seen that :
If C is a norm bounded order closed convex set in L^Φ, then it is $\sigma(L^\Phi, L^\Psi)$-closed.

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If C is a norm bounded order closed convex set in L^Φ, then it is $\sigma(L^\Phi, L^\Psi)$-closed.

Corollary. $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has P if and only if $\sigma(L^\Phi, L^\Psi)$ has *Krein-Smulyan property*, i.e., if C is a convex set such that $C \cap nB_{L^\Phi}$ is $\sigma(L^\Phi, L^\Psi)$-closed for all n, then C is $\sigma(L^\Phi, L^\Psi)$-closed.

Which $\sigma(L^\Phi, L^\Psi)$ has KS property?

Theorem

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1. $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has P.
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3. Either Φ or $\Psi \in \Delta_2$.

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We need: if Φ and $\Psi \notin \Delta_2$, construct a convex set C in L^Φ so that $C \cap nB_{L^\Phi}$ is $\sigma(L^\Phi, L^\Psi)$-closed for all n, $0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)}$ and $0 \notin C$.

A counterexample

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Main idea: If Φ and $\Psi \notin \Delta_2$, then L^Φ contains a lattice isomorphic copy of $\ell^\infty \oplus \ell^1$ and $\sigma(L^\Phi, L^\Psi)$ induces the topology $w^* \oplus w$ on $\ell^\infty \oplus \ell^1$.

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A set S in $\ell^\infty \oplus \ell^1 = \ell^\infty \oplus (\oplus \ell^1)_1$ that contains 0 in its $w^* \oplus w$-closure but no bounded subset does.

$S = \{x_{k,j} : k, j \in \mathbb{N}\}$, where

$$x_{k,j} = (0, \ldots, 0, \overset{j\text{th coord}}{2^k}, 2^k, \ldots) \oplus (0, \ldots, 0, \overset{j\text{th coord}}{\frac{e_j}{2^k}}, 0, \ldots).$$

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Example for C is $C =$ convex hull of S.

Law invariant sets (What if the convex set is nicer?)

A subset S of $L^0(\Omega, \Sigma, \mathbb{P})$ is *law invariant* if $f \in S$, $g \in L^0$, $g \overset{\text{dist}}{=} f$ implies $g \in S$.

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Let C be a law invariant convex subset of L^Φ. Then C is order closed $\iff$ $\sigma(L^\Phi, L^\Psi)$-closed $\iff$ $\sigma(L^\Phi, H^\Psi)$-closed.

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Enough to show order closed $\implies$ $\sigma(L^\Phi, H^\Psi)$-closed.

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$f \in L^\Phi \implies$ there exists a sequence of finite measurable partitions (π_n) of Ω so that $\mathbb{E}[f|\pi_n] \xrightarrow{o} f$.

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C convex, norm closed, law-invariant set in $L^\Phi \implies \mathbb{E}[f|\pi] \in C$ for all $f \in C$ and all finite partition π of Ω.

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$\mathbb{E}[f_\alpha|\pi_n] \to \mathbb{E}[f|\pi_n]$ weakly.
C order closed convex $\implies$ norm closed $\implies$ weakly closed.
So $\mathbb{E}[f|\pi_n] \in C$.

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C order closed. So $f \in C$.

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Jouini, Schachermayer & Touzi (2006)
Let C be a law-invariant convex set in L^∞. Then C is norm closed $\iff$ C is $\sigma(L^\infty, L^1)$-closed. $\iff$ C is order closed.

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Theorem

Every norm closed law-invariant convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed if and only if $\Phi \in \Delta_2$.

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Every norm closed law-invariant convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed if and only if $\Phi \in \Delta_2$.

If $\Phi \notin \Delta_2$, B_{H^Φ} is norm closed law-invariant but not $\sigma(L^\Phi, L^\Psi)$-closed.

Thank You

Proof of last lemma

Lemma

C convex, norm closed, law-invariant set in $L^\Phi \implies \mathbb{E}[f|\pi] \in C$ for all $f \in C$ and all finite partition π of Ω.

Suppose $f \in C$, WLOG $\int \Phi(|f|) \leq 1$.
Given N, choose $b > (N-1)c$ and disjoint sets $A_1, \ldots, A_N$:

$$A_1 = \{|f| > b\} \quad \text{and} \quad \frac{1}{\mathbb{P}((\cup A_i)^c)} \int_{(\cup A_i)^c} f \, d\mathbb{P} \approx \int_\Omega f \, d\mathbb{P}.$$

Construct f_i from f by swapping $f\chi_{A_1}$ with $f\chi_{A_i}$, $1 \leq i \leq N$.
Let $g = \frac{1}{N}\sum_{i=1}^N f_i$. Then $|g| \leq \frac{2}{N}|f_i|$ on A_i and $g = f$ outside $\cup A_i$.
$g = g\chi_{\cup A_i} + f\chi_{(\cup A_i)^c}$.
The first part is dominated by $\frac{2}{N}|f|$ and hence has small norm. he second part belongs to L^∞.
Average with rearrangements until the second part is nearly the constant $\int f \, d\mathbb{P}$.