M-operators on Partially Ordered Banach Spaces

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Outline

Notations

Let X be a normed linear space. We use the following notations:

- $\mathcal{B}(X)$ denotes the set of all bounded linear operators on X.
- The pair (X, K) denotes a partially ordered Banach space X with positive cone K. Unless stated K is assumed to be generating and normal.

Introduction

- Marek and Syzld – M-matrices to partially ordered Banach spaces
- Berman and Plemmon – Characterization of M-matrices
- Koliha and Cain – Characterized positive stability of operators on complex Hilbert spaces
- Characterization of an invertible M-operator
- Generalization of a result on singular M-matrices

Definitions

Definition 1

A square matrix A is called a *Z-matrix* if the off-diagonal entries of A are all non-positive, i.e., $a_{ij} \leq 0$ for $i \neq j$.

Definition 2

A Z-matrix A is called an *M-matrix* if it can be expressed in the form

$$A = sI - B, \quad B \geq 0$$

where $s \geq r(B)$.

Definitions

Definition 3

A square matrix A is called *positive stable* if the real part of each eigenvalue of A is positive.

Definition 4

A square matrix A is said to be *convergent* if $r(A) < 1$.

Motivation

Theorem 5 (Berman and Plemmons, 1994)

Let $A \in \mathbb{C}^{n\times n}$. Then the following are equivalent.

(a) *A is positive stable.*

(b) *There exists a positive definite matrix W such that $AW + WA^*$ is positive definite.*

(c) *$A + I$ is nonsingular and $G = (A + I)^{-1}(A - I)$ is convergent.*

(d) *$A + I$ is nonsingular and for $G = (A + I)^{-1}(A - I)$ there exists a positive definite matrix W such that $W - G^*WG$ is positive definite.*
Additionally, if A is a Z-matrix, then each of the above statements is equivalent to

(e) *A is a nonsingular M-matrix.*

Definitions

Definition 6

A bounded linear operator T on a complex Banach space X is called *positive stable* if the real part of each spectral value of T is positive.

Definition 7

A bounded linear operator T on a complex Hilbert space H is called *positive definite* if $\langle Tu, u\rangle \geq \alpha||u||^2$ for some $\alpha > 0$ and for all $u \in H$.

An extension to infinite dimension

The following theorem is an extension of the four equivalent conditions in Theorem 5 to a general Hilbert space.

Theorem 8 (Koliha, 1973 and Cain, 1973)

Let T be a bounded linear operator on a complex Hilbert space H. Then the following statements are equivalent:

(a') *T is positive stable.*

(b') *There exists a unique positive definite solution X for the equation $T^*X + XT = P$ for each positive definite P.*

(c') *If $-1 \notin \sigma(T)$, then $G = (I+T)^{-1}(I-T)$ is convergent.*

(d') *If $-1 \notin \sigma(T)$, then for $G = (I+T)^{-1}(I-T)$ there exists a unique positive definite solution X for the equation $X - G^*XG = P$ for each positive definite P.*

M-operators

Definition 9

Let (X, K) be a partially ordered Banach space. An operator $T \in \mathcal{B}(X)$ is said to be a Z-*operator* if $T = sI - B$, with $s \geq 0$, $B \geq 0$. A Z-operator is said to be an M-*operator* if $s \geq r(B)$.

Note that the operator T is invertible if and only if $s > r(B)$.

Theorem 10 (Krein-Bonsall-Karlin (KBK))

Let (X, K) be a partially ordered Banach space. Then, for every positive operator $T \in \mathcal{B}(X)$, $r(T)$ belongs to the spectrum of T.

Theorem 11

Let (X, K) be a partially ordered Banach space. Let $T = sI - B \in \mathcal{B}(X)$, where $s \geq 0$ and $B \geq 0$. Then the following statements are equivalent:
(a'') T is positive stable.
(e'') T is an invertible M-operator.
(c'') If $-1 \notin \sigma(T)$, then $G = (I - T)(I + T)^{-1}$ is convergent.

Sketch of the proof:

$(a'') \Rightarrow (e'')$:

- KBK Theorem : $r(B) \in \sigma(B)$
- Spectral mapping theorem: $s - r(B) \in \sigma(T)$
- Positive stability: $r(B) < s$

$(e'') \Rightarrow (a'')$:

- T is an invertible M-operator: $r(B) < s$
- Spectral mapping theorem: spectral values of T are of the form $s - \lambda$ for some $\lambda \in \sigma(B)$

Proof continued

$(a'') \Leftrightarrow (c'')$:

- Spectral mapping theorem: $\lambda \in \sigma(T)$ if and only $\beta \in \sigma(G)$ where $\beta = \dfrac{1-\lambda}{1+\lambda}$
- Simple calculation: $|\beta| < 1$ if and only if $\operatorname{Re}\lambda > 0$
- $|\beta| < 1$ for all $\beta \in \sigma(G)$ i.e. $r(G) < 1$ if and only if $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(T)$ i.e. T is positive stable.

Combining Theorem 8 and Theorem 11, we obtain the following result, which generalizes Theorem 5.

Theorem 12

Let (X, K) be a partially ordered Hilbert space. Let $T = sI - B \in \mathcal{B}(X)$, where $s \geq 0$ and $B \geq 0$. Then the following statements are equivalent:
(a^*) T is positive stable.
(b^*) There exists a unique positive definite solution W for the equation

$$T^*W + WT = P,$$

for each positive definite P.
(c^*) If $-1 \notin \sigma(T)$, then $G = (I - T)(I + T)^{-1}$ is convergent.
(d^*) If $-1 \notin \sigma(T)$, then for $G = (I - T)(I + T)^{-1}$ there exists a unique positive definite solution W for the equation $W - G^*WG = P$, for each positive definite P.
(e^*) T is an invertible M-operator.
(f^*) T^{-1} exists and $T^{-1} \geq 0$.

Theorem 13 (Fan, 1992)

If $A - I$ is an M-matrix, then so is $I - A^{-1}$.

The following result is a generalization of Theorem 13.

Theorem 14

Let (X, K) be a partially ordered Banach space. Let $S \in \mathcal{B}(X)$ be such that $S - I$ is an invertible M-operator. Then S and $I - S^{-1}$ are invertible M-operators.

Idea of the proof:

- KBK theorem and spectral mapping theorem: S is an invertible M-operator.
- Positive stability of an invertible M-operator: $I - S^{-1}$ is an invertible M-operator.

The following example illustrates the above theorem.

Example

Consider the Hilbert space $H = \mathbb{R} \oplus l^2(\mathbb{N}) = \{(\xi, x) : \xi \in \mathbb{R}, x \in l^2(\mathbb{N})\}$ *with the inner product defined in the following way:*
$\langle (\xi_1, x_1), (\xi_2, x_2) \rangle = \xi_1 \xi_2 + \langle x_1, x_2 \rangle$.

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$\langle (\xi_1, x_1), (\xi_2, x_2) \rangle = \xi_1 \xi_2 + \langle x_1, x_2 \rangle$.
Then the norm induced by the inner product is $||(\xi, x)|| = \sqrt{\xi^2 + ||x||^2}$.
Consider the cone $K = \{(\xi, x) \in H : \xi \geq 0, \xi \geq ||x||\}$ *on* H. K *is generating and normal.*

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Consider the cone $K = \{(\xi, x) \in H : \xi \geq 0, \xi \geq ||x||\}$ *on* H. K *is generating and normal.*
Let $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ *be defined by* $B(x) = (t_1 x_1, t_2 x_2, t_3 x_3, \cdots)$*with* $\frac{1}{2} < t_i \leq 1$ *for all* i. *Consider the operator* D *on* H *defined by* $D(\xi, x) := (\xi, B(x))$.

The following example illustrates the above theorem.

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$D(\xi, x) := (\xi, B(x))$.
Then D *is a positive linear operator on* (H, K) *and* $r(D) \leq 1$. *Now define* $T : H \to H$ *by* $T = 2I - D$. *Clearly* T *is an invertible* M*-operator.*

The following example illustrates the above theorem.

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$\langle(\xi_1, x_1), (\xi_2, x_2)\rangle = \xi_1\xi_2 + \langle x_1, x_2\rangle$.
Then the norm induced by the inner product is $||(\xi, x)|| = \sqrt{\xi^2 + ||x||^2}$.
Consider the cone $K = \{(\xi, x) \in H : \xi \geq 0, \xi \geq ||x||\}$ *on* H. K *is generating and normal.*
Let $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ *be defined by* $B(x) = (t_1x_1, t_2x_2, t_3x_3, \cdots)$*with* $\frac{1}{2} < t_i \leq 1$ *for all* i. *Consider the operator* D *on* H *defined by* $D(\xi, x) := (\xi, B(x))$.
Then D *is a positive linear operator on* (H, K) *and* $r(D) \leq 1$. *Now define* $T : H \to H$ *by* $T = 2I - D$. *Clearly* T *is an invertible M-operator. Let* $S = 3I - D$. *Then* $S - I = T$. *Again* S *is an invertible M-operator and* $S^{-1}(\xi, x_1, x_2, \ldots) = (\frac{\xi}{2}, \frac{x_1}{3-t_1}, \frac{x_2}{3-t_2}, \ldots)$ *where* $\frac{2}{5} \leq \frac{1}{3-t_i} \leq \frac{1}{2}$ *for* $i \in \mathbb{N}$. *Now* $\sigma(S^{-1}) = \overline{\{\frac{1}{2}, \frac{1}{3-t_1}, \frac{1}{3-t_2}, \ldots\}}$. *Hence* $r(S^{-1}) = \frac{1}{2}$ *and* $I - S^{-1}$ *is invertible.*

Singular case

Definition 15

Let $A \in \mathbb{R}^{n\times n}$ $(n > 1)$ be non-zero. A is said to be *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices. Otherwise, A is called an *irreducible matrix*.

Definition 16

An M-matrix $A = sI - B$ (where $B \geq 0$ and $0 \neq s \geq r(B)$) is said to have *property c* if the sequence $(s^{-k}B^k)_{k\in\mathbb{N}}$ is convergent. Property c can be defined analogously in the case of a partially ordered Banach space.

Singular M-Matrix

Recall that a Z-matrix $A = sI - B$, where $B \geq 0$, is a *singular M-matrix* if $s = r(B)$.

Theorem 17 (Berman and Plemmons, 1990)

Let $A \in \mathbb{R}^{n \times n}$ $(n > 1)$ be a singular, irreducible M-matrix. Then:
(a) A has rank $n-1$.
(b) There exists a vector $x \in int(\mathbb{R}^n_+)$ such that $Ax = 0$.
(c) $Ax \geq 0 \Rightarrow Ax = 0$.
(d) A has property c.

Preliminary result

Theorem 18 (Marek and Syzld, 1990)

Let (X, K) be a partially ordered Banach space. Let $T = sI - B \in \mathcal{B}(X)$ (where $B \geq 0$ and $s \geq r(B)$) be an M-operator with property c. Then the following implication holds:

$$Tx \in K \text{ and } x \in R(T) \Rightarrow x \in K.$$

Definition 19

Let (X, K) be a partially ordered Banach space. A positive operator $T \in \mathcal{B}(X)$ is called *irreducible* if for every $\alpha > 0$ and $x \in K \setminus \{0\}$ such that $Tx \leq \alpha x$ it follows that x is a quasi-interior point of K.

A vector $x \in K$ is called a *quasi-interior point* of K if $f(x) > 0$ for every non-zero $f \in K^*$.

A Generalization of the result on singular M-matrices

Theorem 20

Let (X, K) be a partially ordered Banach space where $int(K)$ is non-empty. Let $T = sI - B$, where $s > 0$, $s = r(B)$ and B is an irreducible positive operator with $r(B)$ as a pole of the resolvent map $\mathcal{R}_T$. Then the following results hold:

(a') *The dimension of the nullspace of T is 1.*

(b') *There exists a vector $u \in int(K)$ such that $Tu = 0$.*

(c') *If X is reflexive, then $Tx \in K$ implies $Tx = 0$.*

(d') *If X is reflexive, then $Tx \in K, x \in R(T)$ implies $x = 0$.*

Idea of the proof:

(a') B is positive and irreducible : $r(B)$ is a simple eigenvalue of B.
(b') Perron-Frobenius: Existence of positive vector u such that $Bu = r(B)u$.
Irreducibility of B: $u \in int(K)$.
(c')Irreducibility of B^*: $f \in K^* \setminus \{0\}$ is a quasi-interior point such that $T^*f = 0$.
$Tx \in K$ and $Tx \neq 0$. f is a quasi-interior implies $x(T^*f) > 0$, a contradiction since $T^*f = 0$.
(d') Let $Tx \in K$ and $x \in R(T)$. By (c'), $Tx = 0$ and by (a'), $x = \alpha u$ for some $\alpha \in \mathbb{R}$. Using (c') and some simple calculation we can get $\alpha \geq 0$. Similar argument gives $-x \in K$ that is $x \in K \cap (-K) = \{0\}$.

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THANK YOU!