On characterizing the spectra of nonnegative matrices

Thomas J. Laffey
(University College Dublin)

July 2017

Given a list $\sigma = (\lambda_1, \quad ... \quad, \lambda_n)$ of complex numbers, the **nonnegative inverse eigenvalue problem (**NIEP**)** asks for necessary and sufficient conditions for σ to be the spectrum of an (entrywise) nonnegative matrix**.** If σ is the spectrum of a nonnegative matrix *A*, then σ is said to be **realizable** and *A* is referred to as a **realizing matrix** for σ**.**

If, in addition, we require σ to have a diagonalizable realizing matrix, the problem is referred to as DNIEP.

If σ consists of real numbers, the problem is sometimes called RNIEP, and the problem of determining whether σ has a nonnegative symmetric realizing matrix is called the **symmetric nonnegative inverse eigenvalue problem (**SNIEP).

Obvious necessary conditions for realizability are that the Newton power sums

$$s_k := \lambda_1^k + \quad ... \quad + \lambda_n^k$$

are nonnegative, for $k = 1, 2, 3, \quad ...$ Another necessary condition coming from the Perron-Frobenius Theorem is that

$$\rho := \max\{| \lambda_j |: j = 1, \quad ... n\}$$

belongs to σ.
This is called the Perron condition.

An important set of necessary conditions discovered independently by Loewy and London (LAMA **6** (1978) 83-90) and Johnson (LAMA **10** (1981) 113-129) is

$$(\textbf{JLL}\text{ inequalities}) \qquad n^{k-1} s_{km} \geq s_m^k$$

for all positive integers k, m.

Loewy and London (LAMA **6** (1978) 83-90) proved that Conditions (1), (2), (3) are sufficient for realizability for $n \leq 3$ and also for $n = 4$ if all the λ_i are real.

Robert Reams (LAMA **41** (1996) 367-373) proved that for $n = 4$ and $s_1 = 0$, they are also sufficient.

The general case when $n = 4$ was fully resolved in joint work with Eleanor Meehan (ELA **3** (1998) 119-128, thesis NUI Dublin, 1998), but the list of necessary conditions, expressed in terms of the power sums s_i is quite lengthy and are not written down here.

A different, more graph-theoretic approach, to the case $n = 4$ was found by Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuán and Pisonero (LAA **426** (2007) 729-779) but again, the list of necessary conditions, this time expressed in terms of coefficients of polynomials, is too lengthy to include here.

For $n = 5$ and $s_1 = 0$, in collaboration with Meehan (LAA **302-303** (1999) 295-393), we proved:

Theorem. Let $n = 5$ and assume that σ satisfies $s_1 = 0$. Then σ is realizable if and only if

(i) $s_k \geq 0$ for $k = 2, 3, 4, 5$.

(ii) $4s_4 \geq s_2^2$ and

(iii) $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geq 0$.

In terms of n, the NIEP has not yet been fully resolved for the case $n = 5, s_1 > 0$ or for any $n > 5$.

The case of symmetric realizability SNIEP.

Assume that $\sigma = (\lambda_1, \quad ... \quad, \lambda_n)$ is a list of real numbers with

$$\lambda_1 \geq \quad ... \quad \geq \lambda_n.$$

Work of Fiedler (LAA **9** 119-142) and Loewy and London (LAMA **6** (1978) 83-90) shows that if $n \leq 4$ and all the λ_i are real, then σ is symmetrically realizable if and only if it is realizable.
In the case $n = 4$ the necessary conditions simplify to

$$s_1 \geq 0, \ \ \lambda_1 + \lambda_4 \geq 0.$$

For the case $n = 5$, Oren Spector, a student of Raphi Loewy, resolved the problem when $s_1 = 0$.

He proved (LAA **434** (2011) 1000-1017) that if $s_1 = 0$, then σ is symmetrically realizable if and only if

$$\text{(i)} \qquad \lambda_2 + \lambda_5 \leq 0 \;\text{ and }\quad \text{(ii)} \qquad s_3 \geq 0.$$

The necessity of the condition (i) here was established by McDonald and Neumann (AMS Contemp. Math . **259** (2000) 387-407) in the case that the realizing matrix was required to be permutation irreducible and extended to the general case by Loewy and McDonald (LAA **393** (2004) 276-298).

If $\lambda_3 \leq 0$, the sufficiency of the conditions was already established in the Loewy-McDonald paper. The proof of the sufficiency when λ_3 is positive.is very difficult and it involves an elaborate geometric analysis of the inequality $s_3 \geq 0$.

For the case $n = 5$ and $s_1 > 0$, in the paper cited, Loewy and McDonald have made considerable progress.

In this case, the McDonald-Neumann inequality takes the form $\lambda_2 + \lambda_5 \leq s_1$, and, in the case $\lambda_3 \leq 0$, this turns out to be the only necessary new inequality not needed in realizing the same spectrum without requiring the realizing matrix to be symmetric. In the case $s_3 > 0$,they study the {zero, nonzero} pattern of the entries of a putative symmetric nonnegative realizing matrix as a graph, and reduce the problem to considering two explicit graphs.

If $s_1 \geq \lambda_3$, the problem can be resolved, using the results for $n = 4$.

In a computational *tour de force*, Loewy and Spector (LAA **528** (2017) 205-272) have investigated the remaining situation where $s_1 \leq \lambda_3$. They have shown that if $s_1 \geq \lambda_1/2$, then the condition $s_1 \geq \lambda_3$ is necessary for symmetric realizability.

The remaining case where $0 < s_1 < \lambda_1/2$ is still not resolved.

Johnson, Marijuán and Pisonero (LAA **512** (2017) 125-135) have established some non-realizability results in this region; for example, they show that

$$\sigma = (6, 3, 3, -5, -5)$$

is not symmetrically realizable despite satisfying all known general necessary conditions.
It is easy to show that this σ is realizable without requiring the symmetry condition.

Distinguishing between realizability and symmetric realizability (RNIEP v SNIEP) is currently a topic of interest.

The example $\sigma = (3 + t, 3 - t, -2, -2, -2), t \geq 0$, has been much studied.
In particular, in the 1980s, Hartwig and Loewy proved by a complicated argument that it is symmetrically realizable if and only if $t \geq 1$. This is now an easy consequence of the McDonald-Neumann inequality.

With Meehan (ELA **3** (1998) 119-129), we showed that σ is realizable if and only if $t \geq \sqrt{16\sqrt{6} - 39} = 0.4...$.

With Ŝmigoc (LAA 421 (2007) 97-142) we proved that σ with one zero added is symmetrically realizable if $t \geq \frac{1}{3}$.

Lixing Han using lengthy computer calculations has found strong evidence that $t \geq \frac{1}{3}$ is the best bound here and also for the case of σ with two zeros added.

The question of what the diagonal entries of a symmetric realizing matrix of a given spectrum can be, is of great interest.

Without the nonnegativity condition, this problem was solved by Schur.

Fiedler(LAA **9** (1974) 119-142) made contributions to the nonnegative case.

Several interesting new results have been found by Ellard and Ŝmigoc (LAA **498** (2016) 521-552).

Adding zeros to the spectrum

Research on the NIEP was transformed by the appearance in Annals of Mathematics (**133** (1991) 240-316) of the following remarkable result:

Boyle and Handelman Theorem. Let $\sigma = (\lambda_1, \quad ... \quad, \lambda_n)$ be a list of complex numbers satisfying

(i) $\max\{ \mid \lambda_j \mid : j = 1, \quad ..., n\} = \lambda_1 > \mid \lambda_i \mid$, for all $i > 1$. and

(ii) $s_k := \lambda_1^k + \quad ... \quad + \lambda_n^k \geq 0$, for all positive integers k, and $s_k = 0$ implies $s_d = 0$, for all positive divisors d of k.

Then there exists an integer $N \geq 0$ such that

$$(\lambda_1, \quad ... \quad, \lambda_n, 0, \quad ... \quad, 0) \ \ (N \text{ zeros})$$

is the spectrum of a nonnegative $(n + N) \times (n + N)$ matrix.

Both the result itself, and the proof which employs a vast array of tools from algebra, analysis and ergodic theory, is most impressive.

The theorem is primarily an existence theorem and it does not offer a bound on the minimum N required, or a constructive algorithm.

Using the JLL inequalities referred to earlier, it is easy to show that even for $n = 3$, there are spectra σ for which the minimum N required in the theorem can be arbitrarily large.

We now present a constructive version which requires slightly stronger hypotheses.

Let $X_m =$

$$\begin{pmatrix} x_1 & 1 & 0 & . & . & . & 0 \\ x_2 & x_1 & 2 & 0 & . & . & . \\ . & x_2 & x_1 & 3 & 0 & . & . \\ & & . & & & & . \\ . & & & . & . & & 0 \\ x_{m-1} & . & & & . & & m-1 \\ x_m & x_{m-1} & . & & . & x_2 & x_1 \end{pmatrix}$$

where the x_i are commuting indeterminates.

Theorem (L., LAA **436** (2012) 1701-1709). Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be a list of complex numbers satisfying

(i) $\max\{ \mid \lambda_j \mid : j = 1., \quad ... n\} = \lambda_1 > \mid \lambda_i \mid$, for all $i > 1$, and

(ii) $s_k := \lambda_1^k + \quad ... \quad + \lambda_n^k :\geq 0$, for all positive integers k, and $s_k = 0$ implies $s_d = 0$, for all positive integers $d < k$.

Then there exists an integer $N \geq 0$ and a nonnegative specialization of the matrix X_{n+N} with spectrum

$$(\lambda_1, ..., \lambda_n, 0, ..., 0) \text{ (N zeros)}$$

Furthermore, the minimum $N = N(\sigma)$ for which the result holds can be bounded in terms of the λ_i.

The bound is not in general best possible.

The result provides an easy-to use algorithm to construct realizations of σ with several zeros added.

If B is an $r \times r$ matrix with all its entries positive, then the result implies that there is a nonnegative specialization of the matrix X_m with the same nonzero spectrum as B and one can bound the least such m as a function of the spectrum of B.

Boyle and Handelman asked whether there is an analogous result to their theorem for the problem SNIEP.

Jointly with Johnson and Loewy we proved (Proc. AMS 124 (1996) 3647-3651) that the answer is No; in fact, if

$$(\lambda_1, \quad ... \quad, \lambda_n, 0, \quad ... \quad, 0) \ \ (N \text{ zeros})$$

is symmetrically realizable, then

$$(\lambda_1, \quad ... \quad, \lambda_n, 0, \quad ... \quad, 0) \quad (\frac{n(n+1)}{2} \text{ zeros})$$

is symmetrically realizable.

Examples suggest that the bound $\frac{n(n+1)}{2}$ here can be substantially reduced, possibly to below n, and, if proved, this would have many interesting consequences.

In collaboration with Ŝmigoc, we have obtained an analogous result for realizability by diagonalizable nonnegative matrices (DNIEP). Here the bound $\frac{n(n+1)}{2}$ is replaced by $n + n^2$.

Some constructive results

In 1949, in what is considered to be the first result on the NIEP, Suleimanova showed that if all the λ_j are real and only one is positive, then σ is realizable if and only if $s_1 \geq 0$.

In 1977, Friedland (Israel J. Math. **29** (1977) 43-60) showed that a constructive realization using a companion matrix.

A direct generalization of Suleimanova's result is the following:

Theorem (L. and Ŝmigoc, LAA **416** (2006) 148-159). Let $\sigma = (\lambda_1, \quad ... \quad, \lambda_n)$ be a complex-conjugate closed list of complex numbers satisfying $\lambda_1 > 0$ and

$$\mathrm{Re}(\lambda_j) \leq 0, \text{ for } j = 2, \quad ... \quad, n.$$

Then σ is realizable if and only if $s_1 \geq 0, s_2 \geq 0$ and $ns_2 \geq s_1^2$. If σ is realizable, then it has a realizing matrix of the form $(\frac{s_1}{n})I_n + C$, where C is a nonnegative trace zero companion matrix.

A detailed account of the various constructive methods and comparison between them can be found in Borobia, Moro and Soto (LAA **393**, (2004), 73-84, **396** (2005) 223-241).

Marijuán, Pisonero and Soto (LAA **530** (2017) 344-365) contains a comprehensive account of current techniques in the study of SNIEP.

Relationship with power series

Let $\sigma = (\lambda_1, \lambda_2, \quad ... \quad, \lambda_n)$ be a complex conjugate closed list of complex numbers and

$$f(t) = \Pi_{j=1}^{n}(1 - \lambda_j t).$$

Then the formal expansion of the logarithmic derivative, $f'(t)/f(t)$, of $f(t)$ in powers of t is

$$-(s_1 + s_2 t + s_3 t^2 + \quad ... \quad),$$

where

$$s_j = \lambda_1^j + \lambda_2^j + \quad ... \quad + \lambda_n^j.$$

Let k be a positive integer.
If

$$g(t) = 1 + g_1 t + g_2 t^2 + g_3 t^3 + \quad ...$$

satisfies $g^k = f$, then
$g'(t)/g(t) = -(\frac{1}{k})(s_1 + s_2 t + s_3 t^2 + \quad ... \quad),$
So, if the coefficients of the powers of t in the expansion of $g'(t)/g(t)$ are non-positive, then so are those of $f'(t)/f(t)$.
If $g(t)$ has the form above with all $g_i \leq 0$, then it follows immediately that the formal expansion of $g'(t)/g(t)$ has non-positive coefficients.

In a very impressive paper, Kim, Ormes, Roush (JAMS **13** (2000) 773-806) proved a conjecture of Boyle and Handelman on the validity of an analogue of their result for nonzero spectra with corresponding characteristic polynomials having integer coefficients.

In attempting to use their proof as the basis for a constructive algorithm, given f,one is confronted with finding related power series of the form of g above with all $g_j \leq 0$.

In the course of this work, we observed that, with σ, f as above, if σ is realizable, in many cases one could find a positive integer k for which the formal expansion of $g = f^{1/k}$ has all g_j non-positive.

In no case, could we prove that such a k does not exist, though we were able to exclude it happening for k small enough to carry out the computation.

This led to the question:of whether such a k always exists for realizable lists.

A literature search on Taylor series about $z = 0$ having leading coefficient 1 and all other coefficients non-positive did not provide an answer, though such series have been studied by Lamperti and others.

Also, the partial sums of such series (made monic by replacing t by $1/x$) correspond to the characteristic polynomials of nonnegative companion matrices and so have a Perron root, which monotonically increases as more terms of the series are chosen;

The limit points of roots of partial sums of Taylor series have been studied by Jentzsch, Dvoretsky, Edrei and others.

As a test, we considered the case where σ consists of positive real numbers and we obtained the following result:
Proposition.(L., Math. Proc. Royal Irish Acad. (T.T. West Memorial Issue)**113A** (2013) 97-106). Let $a_{1,}a_2, \quad ... \quad , a_n$ be positive real numbers and
$f(t) = (1 - a_1 t)(1 - a_2 t) \quad ... \quad (1 - a_n t)$. Then all coefficients in the Taylor expansion of $f(t)^{1/n} - 1$ about $t = 0$ are non-positive.

Illustration of the proof when $n = 3$.
We may assume that
$f(t) = (1 - at)(1 - (a + b)t)(1 - (a + b + c)t)$, where a, b, c are nonnegative.
‘ Observe that

$$f(t) = (1 - at)^3(1 - \frac{bt}{1 - at})^2(1 - \frac{\frac{ct}{1-at}}{1 - \frac{bt}{1-at}}),$$

so

$$f(t)^{1/3} = (1 - at)(1 - \frac{bt}{1 - at})^{2/3}(1 - \frac{\frac{ct}{1-at}}{1 - \frac{bt}{1-at}})^{1/3}.$$

Now we use the fact that the expansion around $z = 0$ of $(1 - z)^a$has all its coefficients positive if $a < 0$, while all coefficients except the leading term 1 are negative, if $0 < a < 1$. Expanding first the last factor in the product for $f(t)^{1/3}$, observe that each term except the leading term has the form

$$W = -w \left[\frac{\frac{ct}{1-at}}{1 - \frac{bt}{1-at}} \right]^m ,$$

where $w \geq 0$ and $m \geq 1$.
Now, multiplying this term by the first factor $(1 - at)$ of $f(t)$, we observe that the expansion of the "numerator" of W has all its coefficients nonnegative.

Also, the exponent in the second factor of $f(t)^{1/3}$ is less than one, so after multiplying W by this factor, the "denominator" of W still has the term $1 - \frac{bt}{1-at}$ raised to a positive power.so the contribution to the expansion of all terms $(1 - at)(1 - \frac{bt}{1-at})^{2/3}W$ have all coefficients non-positive.
The remaining terms come from the expansion of $(1 - at)(1 - \frac{bt}{1-at})^{2/3}$, and a similar argument yields that, except for the factor $(1 - at)$, they have non-positive coefficients.
Finally $1 - at$ has the desired form.
This argument extends easily to lists of n positive elements.

D. Aharonov (Complex Anal Op.Th. (2017), A. Kovačec (Coimbra preprint. 2016-26) and F. Holland (Math. Proc. Royal Irish Acad. **113A** (2013) 13-19) (all independently), have recently produced different proofs of this result and, interpreting the result as a statement in the context of probability and geometric means, Holland has obtained related results on harmonic means and general measures.
It should be noted that in the proposition, the exponent $1/n$ can be replaced by any positive real number $c < 1/n$. The example $f(t) = (1 - t)^n$ shows that $1/n$ is the least exponent for which the result holds in general.

For given positive real numbers $a_1, a_2, \quad \dots \quad, a_n$ and $f(t) = (1 - a_1 t)(1 - a_2 t) \quad \dots \quad (1 - a_n t)$, the least number $c \geq 1/n$ for which the Taylor coefficients of $f(t)^c - 1$ are non-positive, is a function $c = c(a_{1,} a_2, \quad \dots \quad, a_n)$, and, jointly with Loewy and Ŝmigoc, we have recently shown that $c = \frac{s_2}{s_1^2}$.

It is easy to check that for $c > \frac{s_2}{s_1^2}$, the coefficient of t^2 in the expansion of $f(t)^c - 1$ is positive, so the fact that $c(a_{1,} a_2, \quad \dots \quad, a_n) \leq \frac{s_2}{s_1^2}$ is easy to establish, but proving equality is complicated and uses majorization results for positive symmetric polynomials.

The proposition has been extended to large classes of realizable lists. In this case, the exponent $1/n$ has to be replaced by c for some c with $0 < c < 1$; how to determine the largest such c is currently being investigated.

These results suggested looking at the formal expansion in powers of t of $f_n(t)^{\frac{1}{n}}$, where

$$f_n(t) := \det(I_n - tX_n),$$

where

$$X_n := \left(\begin{array}{ccccccc} x_1 & 1 & 0 & . & . & . & 0 \\ x_2 & x_1 & 2 & 0 & . & . & . \\ . & x_2 & x_1 & 3 & 0 & . & . \\ & & . & & & & . \\ . & & & . & . & & 0 \\ x_{n-1} & . & & & . & & n-1 \\ x_n & x_{n-1} & . & & . & x_2 & x_1 \end{array} \right),$$

where the x_i are commuting indeterminates.
Theorem (L., Loewy and Ŝmigoc, Math. Ann. **364** (2016) 687-707). For $0 < c \leq \frac{1}{n}$, the coefficient of each monomial in $t, x_1, \ \ldots \ , x_n$ occurring in the formal expansion of $f(t)^c - 1$is negative.

Corollary. Let r be a positive integer and B an $r \times r$ matrix with positive entries. Then there exists a positive integer n for which the Taylor expansion of $(\det(1 - tB))^{\frac{1}{n}} - 1$ about $t = 0$ has all its Taylor coefficients negative.
One can find a bound for the least such n as a function of the spectrum of B.