The dual Radon-Nikodym property for finitely generated Banach $C(K)$-modules

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Theorem 1

(Lozanovsky - Lotz) Let X be a Banach lattice. Then the following conditions are equivalent.

1. *X is reflexive.*
2. *X does not contain a copy [a] of either c_0 or ℓ^1.*
3. *X does not contain a copy of either c_0 or ℓ^1 as a sublattice.* [b]

[a] If X and Y are Banach spaces we say that X contains a copy of Y if there is a closed subspace of X linearly isomorphic to Y.

[b] If X and Y are Banach lattices we say that X contains a copy of Y as a sublattice if there is a closed sublattice of X lattice isomorphic to Y.

Theorem 2

(Lozanovsky) Let X be a Banach lattice. Then the following conditions are equivalent.

1. *X is weakly sequentially complete.*
2. *X does not contain a copy of c_0.*
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- Before we state one more result in this direction let us recall the following definition.

Definition 3

A Banach space X is said to have the Radon-Nikodym property (RNP) if for every finite measure space $(\Omega, \Sigma, \lambda)$ and for every bounded linear operator $T : L^1(\lambda) \to X$ there exists a strongly measurable $g \in L^\infty(\lambda, X)$ such that

$$Tf = \int\limits_{\Omega} fgd\lambda, \ f \in L^1(\lambda),$$

where the integral is the Bochner integral.

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Theorem 4

(Lotz) Let X be a Banach lattice. The following conditions are equivalent.

1. *The Banach dual $X^\star$ of X has RNP.*
2. *X does not contain a copy of ℓ^1.*
3. *$X^\star$ does not contain a copy of either c_0 or $L^1[0,1]$ as a sublattice.*

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Theorem 5

(Diestel - Uhl) Let (Ω, Σ, μ) *be a finite measure space,* $1 \leq p < \infty$*, and* X *be a Banach space. Then* $L^p(\mu, X)^\star = L^q(\mu, X^\star)$*, where* $1/p + 1/q = 1$*, if and only if* $X^\star$ *has the Radon-Nikodym property with respect to* μ*.*

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- The statements of Theorems 1, 2, and 4 become false if instead of Banach lattices we consider arbitrary Banach spaces. For Theorems 1 and 2 a counterexample is provided by the famous James' space. However the James' space, being quasi-reflexive does have the dual RNP. Therefore as a counterexample in the case of Theorem 4 we need to use another example of James where he constructed a separable Banach space X not containing a copy of ℓ^1 and such that $X^\star$ is not separable. It follows from a later result of Stegall that the space X does not have the dual RNP.

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- But, if instead of the class of all Banach spaces we consider the much smaller classes of finitely generated Banach $C(K)$-modules or Banach $C(K)$-modules of finite multiplicity, which while not contained in the class of all Banach lattices can be considered as its nearest relatives, the analogues of Theorems 1 - 4 remain true.

Definition 6

Let K be a compact Hausdorff space and X be a Banach space. We say that X is a Banach $C(K)$-module if there is a continuous unital algebra homomorphism m of $C(K)$ into the algebra $L(X)$ of all bounded linear operators on X.

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- Because $\ker m$ is a closed ideal in $C(K)$ by considering, if needed, $C(\tilde{K}) = C(K)/\ker m$ we can and will assume without loss of generality that m is a contractive homomorphism and $\ker m = 0$. Then it can be proved that m is an isometry. Moreover, when it does not cause any ambiguity we will identify $f \in C(K)$ and $m(f) \in L(X)$.

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- The following proposition was proved by Veksler in the case when the compact space K is extremally disconnected and in full generality by Hadwin and Orhon.
- **Proposition.** Let X be a Banach $C(K)$-module, $x \in X$, and $X(x)$ be the corresponding cyclic subspace. Then, endowed with the cone $X(x)_+ = cl\{fx : f \in C(K), f \geq 0\}$ and the norm inherited from X, $X(x)$ is a Banach lattice with positive quasi-interior point x.

- **Proposition.** (Orhon) The center $Z(X(x))$ of the Banach lattice $X(x)$ is isometrically isomorphic to the weak operator closure of $m(C(K))$ in $L(X(x))$.

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Definition 8

Let X be a Banach $C(K)$-module. We say that X is finitely generated if there are an $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$ such that the set $\sum_{i=1}^{n} X(x_i)$ is dense in X.

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- **Proposition.** (Orhon) The center $Z(X(x))$ of the Banach lattice $X(x)$ is isometrically isomorphic to the weak operator closure of $m(C(K))$ in $L(X(x))$.
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- Introduction of the second main object of this talk requires some additional preliminaries.

Definition 9

Let X be a Banach space and $\mathcal{B}$ be a Boolean algebra of projections on X. The algebra $\mathcal{B}$ is called Bade complete if
(1) $\mathcal{B}$ is a complete Boolean algebra.
(2) Let $\{\chi_\gamma\}_{\gamma \in \Gamma}$ be an increasing net in $\mathcal{B}$, χ be the supremum of this net, and $x \in X$. Then the net $\{\chi_\gamma x\}$ converges to χx in norm in X.

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Definition 10

Let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on X. $\mathcal{B}$ is said to be of *uniform multiplicity* n, if there exist a set of nonzero pairwise disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ with $\sup e_\alpha = 1$ such that for any e_α and for any $e \in \mathcal{B}$, $e \leq e_\alpha$ the $C(K)$-module eX has exactly n generators.

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- When $\mathcal{B}$ is of uniform multiplicity 1 we have the following result.

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Theorem 11

(Rall) Let $\mathcal{B}$ be of uniform multiplicity one on X. Then X may be represented as a Banach lattice with order continuous norm such that $\mathcal{B}$ is the Boolean algebra of band projections on X.

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Definition 12

A Bade complete Boolean algebra of projections $\mathcal{B}$ on X is said to be of finite multiplicity on X if there exists a collection of disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ such that, for each α, $e_\alpha X$ is n_α-generated and $\sup e_\alpha = 1$.

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- The collection $\{n_\alpha\}$ of positive integers in Definition 12 need not be bounded.

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Definition 13

A Banach $C(K)$-module X is said to be of finite multiplicity (of uniform multiplicity n) if the Boolean algebra of idempotents in $C(K)$ is of finite multiplicity (respectively of uniform multiplicity n) on X.

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- The following theorem describes the structure of Banach $C(K)$-modules of finite multiplicity.

Theorem 14

(Bade). Let X be a Banach $C(K)$-module of finite multiplicity. Then there exists a sequence of disjoint idempotents $\{e_n\}$ in $\mathcal{B}$ such that, for each n, $\mathcal{B}$ is of uniform multiplicity n on e_nX and $\sup e_n = 1$. Also the norm closure of the sum of the sequence of the spaces $\{e_nX\}$ is equal to X.

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Theorem 15

Let X be a finitely generated Banach $C(K)$-module or a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.

1. *X is reflexive.*
2. *X does not contain a copy of either c_0 or ℓ^1.*
3. *Every cyclic subspace of X does not contain a copy of either c_0 or of ℓ^1.*
4. *Every cyclic subspace of X, represented as a Banach lattice, does not contain a copy of either c_0 or of ℓ^1 as a sublattice.*

Theorem 16

. Let X be a finitely generated Banach $C(K)$-module or a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.

1. *X is weakly sequentially complete.*
2. *X does not contain a copy of c_0.*
3. *Every cyclic subspace of X does not contain a copy of c_0.*
4. *Every cyclic subspace of X, represented as a Banach lattice, does not contain a copy of c_0 as a sublattice.*

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Theorem 17

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(1) $X^\star$ has the Radon-Nikodym property.
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Theorem 17

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(1) $X^\star$ has the Radon-Nikodym property.
(2) X does not contain any copy of ℓ^1.

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- We do not know if, under conditions of Theorem 17, the condition that every cyclic subspace of X does not contain a copy of ℓ^1 is sufficient for $X^\star$ to have RNP. But we can prove it if we put additional constraints on Banach $C(K)$-module X.

Theorem 18

Let K be a hyperstonian compact space and let X be a finitely generated Banach $C(K)$-module such that the algebra $\mathcal{B}$ of the idempotents in $C(K)$, is a Bade complete Boolean algebra of projections on X. Then the following conditions are equivalent

1. *$X^\star$ has the Radon-Nikodym property.*
2. *X does not contain any copy of ℓ^1.*
3. *No cyclic subspace of X contains a copy of ℓ^1.*
4. *No cyclic subspace of X, when represented as a Banach lattice, contains a copy of ℓ^1 as a sublattice.*

- The last result can be extended to Banach $C(K)$-modules of finite multiplicity.

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Theorem 19

Let X be a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.
(1) $X^\star$ has RNP.
(2) X does not contain a copy of ℓ^1.
(3) Any cyclic subspace of X does not contain a copy of ℓ^1.
(4) Any cyclic subspace of X represented as a Banach lattice does not contain ℓ^1 as a sublattice.

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- **Final Remarks.**

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- As it is the case with reflexivity and weak sequential completeness, it is easy to see that condition (3) in Theorem 18 cannot be changed to a weaker condition: there is a system of generators $\{x_1, \dots, x_n\}$ such that any cyclic subspace $X(x_i), i = 1, \dots, n$ does not contain a copy of ℓ^1.

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- As it is the case with reflexivity and weak sequential completeness, it is easy to see that condition (3) in Theorem 18 cannot be changed to a weaker condition: there is a system of generators $\{x_1, \dots, x_n\}$ such that any cyclic subspace $X(x_i), i = 1, \dots, n$ does not contain a copy of ℓ^1.
- It is not difficult to produce examples of finitely generated Banach $C(K)$-modules that do not allow a structure of Banach lattice compatible with its structure as a module.
  Still, we do not know any example of a finitely generated Banach $C(K)$-module which is either reflexive, or weakly sequentially complete, or has dual RNP, but is not linearly isomorphic to a closed subspace of a Banach lattice with order continuous norm.

- There is an example of a Banach $C(K)$-module X with the following properties.
  1. X is of uniform multiplicity n, $n > 1$.
  2. X is not separable.
  3. Every cyclic subspace of X is separable and has separable dual. In particular, X cannot be finitely (or even countably) generated.
  4. There are cyclic subspaces of X that are not weakly sequentially complete.

  Thus, while Theorem 18 cannot be applied, by Theorem 19 X has dual RNP.