The Bishop-Phelps-Bollobás property for bilinear forms

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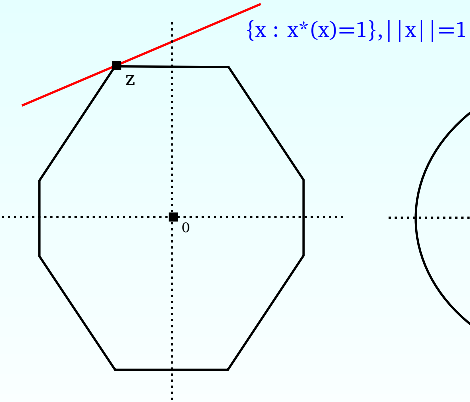
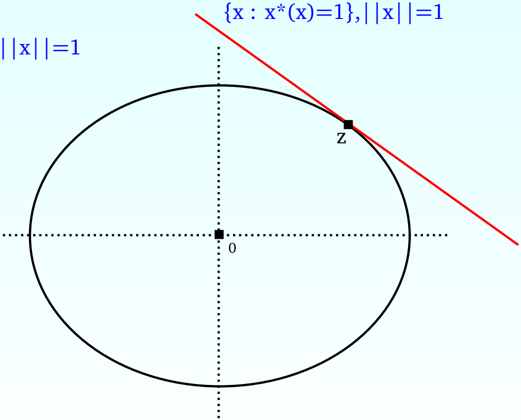
0. Back grounds

Consider finite dimensional space X.





X, X_i, Y : Banach space.

B_X : Closed unit ball of X.

S_X : Closed unit sphere of X.

$\mathcal{L}(X_1, ..., X_n; Y)$: Banach space of all continuous n-linear mappings from $X_1 \times ... \times X_n$ into Y.

Definition

We say that an n-linear mapping $T \in \mathcal{L}(X_1, ..., X_n; Y)$ *attains its norm* if there exists a point $x = (x_1, ..., x_n) \in S_{X_1} \times ... \times S_{X_n}$ such that $\|T(x)\| = \|T\| = \sup\{\|T(z)\| \ : \ z \in B_{X_1} \times ... \times B_{X_n}\}$.
$NA(\mathcal{L}(X_1, \ldots, X_N; Y))$: the set of all norm attaining multilinear mappings.

If a Banach space is finite dimensional, then every functionals attains its norm.

Fact

If a Banach space is reflexive, then every functionals attains its norm.

For arbitrary Banach space? No!

Let

$$x^* = \left(\frac{1}{2^i}\right)_{i=1}^{\infty} \in \ell_1 (= c_0{}^*).$$

For every $x = (x_i)_{i=1}^{\infty} \in B_{c_0}$,

$$x^*(x) = \sum_i \frac{1}{2^i} x_i < \sum_i \frac{1}{2^i} = \|x^*\|.$$

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Example Let us consider

$$x^* = (a_1, a_2, a_3, \dots, a_n, \dots) \in c_0^* = \ell_1.$$

Then, for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ so that

$$\sum_{i>N} |a_i| < \epsilon.$$

Set

$$y^* = (a_1, a_2, a_3, \dots, a_N, 0, 0, 0 \dots). \text{ Then, } \|x^* - y^*\| < \epsilon.$$

This functional attains its norm at

$$\big(sign(a_1), sign(a_2), sign(a_3), \dots, sign(a_n), 0, 0, 0 \dots\big) \in c_0$$

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E. Bishop, R.R. Phelps(1961) For every Banach space X, the set of norm attaining functionals is dense in its dual space X^*.
$(\overline{NA(\mathscr{L}(X;\mathbb{K}))} = \mathscr{L}(X;\mathbb{K}))$

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   X :reflexive $\Longrightarrow \forall Y\ \overline{NA(\mathscr{L}(X;Y))} = \mathscr{L}(X;Y)$
   Y : property $(\beta) \Longrightarrow \forall X\ \overline{NA(\mathscr{L}(X;Y))} = \mathscr{L}(X;Y)$.
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1. Bishop-Phelps-Bollobás Theorem

Theorem

B. Bollobás(1970) *For an arbitrary* $\epsilon > 0$, *if* $x \in B_X$ *and* $x^* \in S_{X^*}$ *satisfy* $|1 - x^*(x)| < \frac{\epsilon^2}{4}$, *then there are* $y \in S_X$ *and* $y^* \in S_{X^*}$ *such that* $y^*(y) = 1$, $\|y - x\| < \epsilon$ *and* $\|y^* - x^*\| < \epsilon$.

Question

1. Can we extend Bishop-Phelps-Bollobás Theorem to operator space between Banach spaces?
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Definition

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) Let X and Y be real or complex Banach spaces. We say that the couple (X,Y) has the Bishop-Phelps-Bollobás property for operators (*BPBP*), if given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$ such that for $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions :

$$\|Su_0\| = 1, \|x_0 - u_0\| < \beta(\epsilon) \text{ and } \|T - S\| < \epsilon$$

1. The couple (X,Y) has the the *BPBP* for finite dimensional Banach spaces X and Y.
2. If Y has property (β), then the couple (X,Y) has the *BPBP* for every Banach space X.

2. The Bishop-Phelps-Bollobás property for operators on ℓ_1 and c_0

Definition

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) A Banach space X is said to have the *AHSP* if for every $\epsilon > 0$ there exists $0 < \eta < \epsilon$ such that for every sequence $(x_k) \subset S_X$ and for every convex series $\sum_{n=1}^{\infty} \alpha_k$ with

$$\Big\| \sum_{n=1}^{\infty} \alpha_k x_k \Big\| > 1 - \eta$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\} \subset S_X$ satisfying

1. $\sum_{k \in A} \alpha_k > 1 - \epsilon$
2.
   1. $\|z_k - x_k\| < \epsilon$ for all $k \in A$
   2. $x^*(z_k) = 1$ for a certain $x^* \in S_{X^*}$ and all $k \in A$

The following Banach spaces have the *AHSP*:

1. a finite dimensional space
2. Lush space (ex. $L_1(\mu)$ and $C(K)$)
3. a uniformly convex space

Theorem

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) The couple (ℓ_1, Y) has the BPBP if and only if Y has the AHSP.

Tool : Representation of operator from ℓ_1 to Y.

$$T : \ell_1 \longrightarrow Y \text{ can be identified with } (y_i)_{i=1}^{\infty} \text{ where } y_i = Te_i.$$

$$Tz = \sum z_i Te_i, \; z = (z_i)$$

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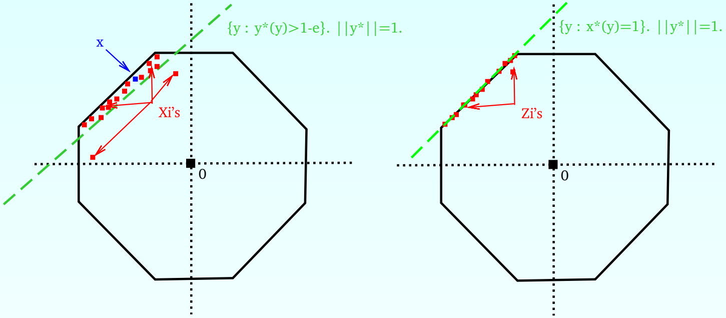
Picture of the AHSP.



Picture 1 : $x = \sum_i \alpha_i x_i$, and $y^*(x) > 1 - \eta$.
Picture 2 : $\|z_i - x_i\| < \epsilon$, and $x^*(z_i) = 1$
for $i \in A$ with $\sum_{k \in A} \alpha_k > 1 - \epsilon$

Definition

For every $\epsilon \in (0,2]$, the *modulus of convexity* of a Banach space $(X, \|\cdot\|)$ is defined by

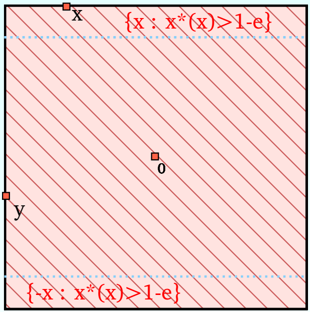
$$\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ : \ x, y \in B_X, \|x-y\| > \epsilon\}.$$

A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

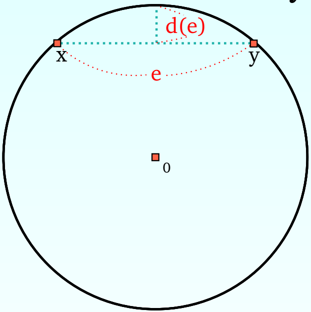
Definition

A Banach space X is said to be *lush* if for every $x, y \in S_X$ and for every $\epsilon > 0$ there is a slice $S = S(B_X, x^*, \epsilon) \subset B_X$, $x^* \in S_{X^*}$, such that $x \in S$ and $dist(y, aconv(S)) < \epsilon$, where $S(B_X, x^*, \epsilon) = \{x \in B_X \ : \ \textbf{Re}\, x^*(x) > 1 - \epsilon\}$.

Lushness



Uniform Convexity



Study about the *BPBP* on c_0.

Problem : Which space Y satisfy that (c_0, Y) has BPBP?

- S.K. Kim (2013) The couple of Banach spaces (c_0, Y) has the *BPBP* for uniformly convex Y.
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- How to describe an operator from c_0 to Y?
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Definition

Let $X_1,\ldots,X_N$ and Y be Banach spaces. We say that $(X_1,\ldots,X_N;Y)$ has the *Bishop-Phelps-Bollobás property for multilinear mappings* (BPBP for multilinear mappings, for short) if given $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that whenever $A\in\mathcal{L}(X_1,\ldots,X_N;Y)$ with $\|A\|=1$ and $\left(x_1^0,\ldots,x_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ satisfy

$$\left\|A\left(x_1^0,\ldots,x_N^0\right)\right\|>1-\eta(\varepsilon),$$

there are $B\in\mathcal{L}(X_1,\ldots,X_N;Y)$ with $\|B\|=1$ and $\left(z_1^0,\ldots,z_N^0\right)\in S_{X_1}\times\ldots\times S_{X_N}$ such that

$$\left\|B\left(z_1^0,\ldots,z_N^0\right)\right\|=1,\quad \max_{1\le j\le N}\|z_j^0-x_j^0\|<\varepsilon \quad\text{and}\quad \|B-A\|<\varepsilon.$$

Known results :

Theorem

R.M. Aron, C. Finet, E. Werner (1995) $\forall i \in \{1,\ldots,n\}$ X_i : *Radon-Nikodým property* $\Longrightarrow \overline{NA(\mathscr{L}(X_1,\ldots,X_n;\mathbb{K}))} = \mathscr{L}(X_1,\ldots,X_n;\mathbb{K})$

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In the same paper, the spaces Y such that $(\ell_1,Y;\mathbb{K})$ has the *BPBP* for bilinear mappings had been characterized.

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versions of Bishop-Phelps and Bollobás theorem

- $\forall Y \quad \overline{NA(\mathscr{L}(\ell_1;Y))} = \mathscr{L}(\ell_1;Y)$
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Y.S. Choi (1997) $NA(\mathscr{L}(L[0,1],L[0,1];\mathbb{K}))$ *is not dense in* $\mathscr{L}(L[0,1],L[0,1];\mathbb{K})$.

- $(\ell_1;Y)$ has BPBp for some Y (with AHSP)

Theorem

Y.S. Choi, H.G. Song (2011) $(\ell_1,\ell_1;\mathbb{K})$ *does not have BPBP for bilinear forms.*

- $\mathscr{L}(\ell_1;\ell_\infty) \simeq \mathscr{L}(\ell_1,\ell_1;\mathbb{K})$
- BPBp for operators holds for $(\ell_1;\ell_\infty)$.

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A bilinear form $B \in \mathscr{L}(\ell_1, \ell_1; \mathbb{K})$ can be represented by $(\alpha_{i,j})_{(i,j)\in\mathbb{N}}$

$$B(x,y) = \sum_{(i,j)\in\mathbb{N}} x_i y_i \alpha_{i,j},\ x = (x_i), y = (y_i), \alpha_{i,j} = B(e_i, e_j)$$

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1. $\|B\| = 1$ and $B(\mathbf{a}_n, \mathbf{a}_n) = 1 - \frac{1}{2n^2}$.

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$\|S\| = |S(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})|$ for some $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$ in S_{l_1} and $\|T - S\| < 1/2$
Then we have either $\|\mathbf{a}_n - \tilde{\mathbf{a}}\| \ge 1/2$ or $\|\mathbf{a}_n - \tilde{\mathbf{b}}\| \ge 1/2$.

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- $\exists Y$ such that $\overline{NA(\mathscr{L}(c_0;Y))} \neq \mathscr{L}(c_0;Y)$

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Note

- ℓ_1 is complex uniformly convex
- (c_0, Y) has BPBP for operators whenever Y is complex uniformly convex.

The modulus of complex convexity H_X for a Banach space X is define by, for $\varepsilon \geq 0$,

$$H_X(\varepsilon) = \inf\left\{ \sup_{0\leq\theta\leq2\pi} \|x + e^{i\theta}y\| - 1 : x \in S_X, \|y\| \geq \varepsilon \right\}.$$

A complex Banach space is said to be uniformly complex convex if $H_X(\varepsilon) > 0$ for all $\varepsilon > 0$.

Lemma

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Let Y be a uniformly complex convex space, L a locally compact Hausdorff space, and A a Borel set of L. For given $0 < \lambda < 1$, if $T \in S_{\mathcal{L}(C_0(L),Y)}$ satisfy that $\|T^{**}P_A\| > 1 - \frac{H_Y(\lambda)}{1+H_Y(\lambda)}$, then $\|T^{**}(I - P_A)\| \leq \lambda$.

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Theorem

Let $X, X_1, \ldots, X_N$ and Y be finite dimensional Banach spaces. Then

(i) $(X_1, \ldots, X_N; Y)$ *has the BPBp for multilinear mappings,*

(ii) $({}^N X; Y)$ *has the BPBp for symmetric multilinear mappings and*

(iii) $(X; Y)$ *has the BPBp for N-homogeneous polynomials.*

Theorem

M. D. Acosta, J. Becerra-Guerrero, D. García and M. Maestre (2013) For the infinite dimensional Banach space $L_1(\mu)$, $(L_1(\mu), L_1(\mu); \mathbb{K})$ does no have the BPBp for bilinear mappings.

Thank you for listening!

presented by Sun Kwang Kim.