Preferences on Choice Sets

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Structure of the Talk

- Background
  - Decision theory
  - Theory of large games
    - Large non-anonymous games
    - Large anonymous games
  - Walrasian general equilibrium theory
- Eilenberg-Sonnenschein research program. Results with Metin Uyanik.

Decision Theory

1. T is a space of states,
2. A is a space of consequences
3. A^T is the space of acts,
4. $\succeq$ be a binary relation on A^T.

Savage's problem: Find assumptions on $\succeq$ that guarantee, and are guaranteed by, the existence of a finitely-additive (subjective) probability μ on T and a real-valued (utility) function u on A such that

$$f \succeq g \iff \int_S u(f(t))d\mu(t) \geq \int_S u(g(t))d\mu(t).$$

Remark: Anscombe-Aumann reformulate the question by considering binary relations on functions from T to probability measures (lotteries) $\mathcal{M}(A)$ on A. This is to say binary relations on $\mathcal{M}(A)^T$.

Large Non-Anonymous Games

A *non-anonymous (individualized) game* $\mathcal{G}$ is an element of $\mathrm{Meas}(T, \mathcal{U})$ where

1. $(T, \mathcal{T}, \lambda)$ is a probability space of players,
2. A is a compact space of actions,
3. $\mathcal{M}(A)$ is the space of probability measures on A endowed with the weak* topology,
4. u is a continuous function on $A \times \mathcal{M}(A)$,
5. $\mathcal{U}$ the space of payoff functions u endowed with its Borel σ-algebra $\mathcal{B}(\mathcal{U})$ generated by the sup-norm topology.

We shall also denote $\mathcal{G}(t)$ by u_t, and since one can always rescale the payoffs, we assume that there is $M > 0$ such that for all $t \in T$, $\|u_t\| \leq M$.

Theorem 1 [**Schmeidler**] *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and $\mathcal{G}$ a large non-anonymous game with a finite action set A. Then there exists a measurable function $f : T \longrightarrow A$ such that for λ-almost all $t \in T$,*

$$u_t(f(t), \lambda \circ f^{-1}) \geq u_t(a, \lambda \circ f^{-1}) \text{ for all } a \in A.$$

Remark: If A has a linear structure on it, then there is a straightforward reformulation of the above result in terms of the integral rather than the law of the function f.

Large Non-Anonymous Games: A Purification Result

A *mixed strategy profile* g (respectively a *pure strategy profile* g^*) is an element of $\mathrm{Meas}(T, \mathcal{M}(A))$.

A *pure strategy profile* g^* is an element of $\mathrm{Meas}(T, A)$.

Theorem 2 *Any mixed strategy equilibrium g for the game $\mathcal{G}$ has a purification.*

Anonymous Games

An *anonymous (distributionalized) game* is a probability measure μ in $\mathcal{M}(\mathcal{U})$.

An anonymous game is said to be *dispersed* if μ is atomless.

An *equilibrium* τ of the game μ is an element of $\mathcal{M}(A \times \mathcal{U})$ with marginal measures τ_A and $\tau_{\mathcal{U}}$ such that

1. $\tau_{\mathcal{U}}$ is μ,
2. $\tau(B_\tau) = \tau(\{(u, a) \in (\mathcal{U} \times A) : u(a, \tau_A) \geq u(x, \tau_A) \text{ for all } x \in A\}) = 1$.

An equilibrium τ can be *symmetrized* if there exist $h \in \text{Meas}(\mathcal{U}, A)$ and another equilibrium τ^s such that $\tau_A = \tau^s_A$ and $\tau^s(\text{Graph}_h) = 1$, where $\text{Graph}_h = \{(u, h(u)) \in (\mathcal{U} \times A) : u \in \mathcal{U}\}$. In this case, τ^s is said to be a *symmetric equilibrium.*

Large Anonymous Games: A Symmetrization Result

Theorem 3 *Every anonymous game* μ *has an equilibrium.*

Theorem 4 *Let* μ *be a dispersed anonymous game such that* A *is a finite set. Then there exists a symmetric equilibrium.*

Theorem 5 *Every equilibrium of a dispersed large anonymous game* μ *can be symmetrized with a countable action set* A*.*

Corollary 1 *A symmetric Cournot-Nash equilibrium distribution exists for a game* μ *with action set* A *whenever* μ *is atomless and* A *is countable.*

Topological Connectedness and Behavioral Assumptions on Preferences: A Two-Way Relationship

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Eilenberg-Sonnenschein Research Program:

Eilenberg 1941

Sonnenschein 1965

Schmeidler 1971

Utility Representation:

Eilenberg 1941

Debreu 1954

Rader 1963

Binary Relations

Let X be a set. A *binary relation* R on X is a subset $R \subset X \times X$. Define

$$\begin{array}{lcl} R^{-1} & = & \{(x,y) \mid (y,x) \in R\}, \\ R(x) & = & \{y \mid (x,y) \in R\}, \\ R^{-1}(x) & = & \{y \mid (y,x) \in R\}, \end{array}$$

where

R^{-1} denote the *transpose* of R,

$R(x)$ the *upper section* of R at x and

$R^{-1}(x)$ the *lower section* of R at x.

Let

$\Delta = \{(x,x) | x \in X\}$ and

R^c denote the complement of R.

Properties of Binary Relations

Let R be a binary relation on a set X and define $I = R \cap R^{-1}$ and $P = R \backslash R^{-1}$. Then, R is

reflexive if $\Delta \subset R$,

complete if $X \times X = R \cup R^{-1}$,

symmetric if $R = R^{-1}$,

asymmetric if $R \cap R^{-1} = \emptyset$,

nontrivial if $R \neq \emptyset$,

transitive if $R^{-1}(x) \times R(x) \subset R$ for all $x \in X$,

negatively transitive if R^c is transitive,

semitransitive if $P^{-1}(x) \times I(x) \subset P$ and $I^{-1}(x) \times P(x) \subset P$ for all $x \in X$.

A topological space X is *connected* if it is not the union of two nonempty, disjoint open sets. A subset of X is connected if it is connected as a subspace.

Theorem (**Eilenberg**)

If X is a connected topological space, then every complete and antisymmetric binary relation on it with closed sections is transitive.

Theorem (**Sonnenschein**)

(a) *If X is a connected topological space, then every complete and semitransitive binary relation on it with closed sections is transitive.*

(b) *If X is a connected topological space, then every complete binary relation on it with closed sections such that its symmetric part is transitive with connected sections, is transitive.*

Theorem (**Schmeidler**)

If X is a connected topological space, then every transitive binary relation on it with closed sections such that its asymmetric part is nontrivial with open sections, is complete.

Theorem (1)

Let X be a topological space and R denote a binary relation on it with symmetric part I and asymmetric part P. Then the following are equivalent.

(a) *X is connected.*

(b) *Every R that is antisymmetric with closed sections, and whose P is nontrivial with open sections, is complete and transitive.*

(c) *Every R that is semitransitive with closed sections, and whose I is transitive, and whose P is nontrivial with open sections, is complete and transitive.*

(d) *Every R that has closed sections, and whose I is transitive with connected sections, and whose P is nontrivial with open sections, is complete and transitive.*

(e) *Every R that is transitive with closed sections, and whose P is nontrivial with open sections, is complete.*

Proof

A Weakening of Connectedness and Eilenberg-Sonnenschein

A topological space X is *connected* if it is not the union of two nonempty, disjoint open sets.

A *component* of a topological space is a maximal connected set in the space, that is, a connected subset which is not properly contained in any connected subset.

A topological space is *k-connected* if it has at most k components.

1-connectedness is equivalent to connectedness

Any k-connected space is l-connected for all $l \geq k$.

Nontriviality

R is a binary relation on a topological space X and $\mathcal{C} = \{C_k\}_{k \in K}$ denote the collection of the components of X.

R is called *nontrivial* if there exists $x, y \in X$ such that $(x, y) \in R \cup R^{-1}$.

R is called $|K|$*-nontrivial* if or all components C, C' of X, there exists $x \in C$ and $y \in C'$ such that $(x, y) \in R \cup R^{-1}$.

> For a connected space, the nontriviality and 1-nontriviality are equivalent.
>
> nontriviality within and across the components.

For $\ell \leq |K|$, R is called ℓ-nontrivial if there exist subcollections $\mathcal{C}^1 = \{C_1^1, \ldots, C_\ell^1\}$ and $\mathcal{C}^2 = \{C_1^2, \ldots, C_\ell^2\}$ of $\mathcal{C}$ such that for all $i, j \leq \ell$, there exists $(x, y) \in (C_i^1 \times C_j^2) \cup (C_j^1 \times C_i^2)$ such that $(x, y) \in R$.

> For $\ell = |K|$, ℓ-nontriviality and $|K|$-nontriviality are equivalent.

Theorem (2)

Let X be a topological space and R denote a binary relation on it with symmetric part I and asymmetric part P. Then the following are equivalent.

(a) *X is 2-connected.*

(b) *Every R that is complete and antisymmetric with closed sections, is transitive.*

(c) *Every R that is complete and semitransitive with closed sections, is transitive.*

(d) *Every R that is antisymmetric with closed sections, and whose P is 2-nontrivial with open sections, is complete and transitive.*

(e) *Every R that is semitransitive with closed sections, and whose I is transitive, and whose P is 2-nontrivial with open sections, is complete and transitive.*

(f) *Every R that has closed sections, and whose I is transitive with connected sections, and whose P is 2-nontrivial with open sections, is complete and transitive.*

Proof

Theorem (3)

Let X be a topological space and k be a positive integer. Then the following are equivalent.

(a) *X is k-connected.*

(b) *Every R that is that is antisymmetric with closed sections, and whose P is k-nontrivial with open sections, is complete.*

(c) *Every R that is semitransitive with closed sections, and whose I is transitive, and whose P is k-nontrivial with open sections, is complete.*

(d) *Every R that has closed sections, and whose I is transitive with connected sections, and whose P is k-nontrivial with open sections, is complete.*

Proof

An Example of a Nontransitive Binary Relation

R is not necessarily transitive for $k > 2$

$X = (0,1) \cup (1,2) \cup (2,3)$ endowed with Euclidean metric

X is 3-connected

Let R be an asymmetric binary relation defined as follows: $(x, y) \in R$ if $x, y \in C_k, x \leq y$, if $x \in (0,1)$ and $y \in (1,2)$, if $x \in (1,2)$ and $y \in (2,3)$, and if $x \in (2,3)$ and $y \in (0,1)$

R is complete and has closed sections

R is nontransitive

Notions of Transitivity

Let R be a relation on a set X, I denote its symmetric part and P denote its asymmetric part.

T : denote R is transitive,

NT : denote P is negatively transitive,

PP : denote P is transitive,

II : denote I is transitive,

PI : denote $P^{-1}(x) \times I(x) \subset P$ for all $x \in X$,

IP : denote $I^{-1}(x) \times P(x) \subset P$ for all $x \in X$.

Theorem (4)

Let R be a binary relation on a set X such that I and P denote its symmetric and asymmetric parts, respectively. Then,

(a) *PP is independent of PI, IP, II, severally and collectively,*

(b) *T is independent of NT,*

(c) $T \Leftrightarrow PP, PI, IP, II$,

(d) $NT \Rightarrow PP, PI, IP$,

(e) $NT\&II \Rightarrow T$,

(f) *if X is a connected topological space and the sections of R are closed and of P are open, then $PI\&IP \Rightarrow NT$, $T \Rightarrow NT$, $PI\&IP\&II \Rightarrow T$,*

(g) *if X is a connected topological space and the sections of I are connected, of R are closed and of P are open, then $II \Rightarrow PI\&IP$.*

Proof

Disontinuous Binary Relations

Let X be a topological space, R be a binary relation on it and P denote its asymmetric part. R is *nonsatiated* in $A \subset X$ if $P(x) \neq \emptyset$ for all $x \in A$.

A subset A of X is called *R-bounded above* if there exists $y \in X$ such that $y \in \cap_{x \in A} R(x)$.

> **(A1)** R has closed upper sections, P has open upper sections, and there exists $\bar{x} \in X$ such that $P(\bar{x}) \neq \emptyset$ and R is nonsatiated in $P(\bar{x})$.
>
> **(A2)** R has closed upper sections, P has open upper sections, and there exists $\bar{x} \in X$ such that $P(\bar{x}) \neq \emptyset$ and every two-element subset of $P(\bar{x})$ is R-bounded above.

Theorem (5)

Let R be a binary relation on a connected topological space X such that its symmetric part is transitive and its asymmetric part is negatively transitive. Then, R is complete and transitive if R or R^{-1} satisfies either (A1) or (A2).

Further Equivalence Results: Definitions

A binary relation R on a topological space X is *fragile* if there exist $x, y \in X$ such that

(i) $(x, y) \in R \backslash R^{-1}$,

(ii) every open neighborhood of (x, y) contains $(x', y') \notin R \cup R^{-1}$.

An asymmetric binary relation P on a topological space X has a *continuous representation* if there exist two continuous real valued functions u and v on X such that for all $x, y \in X$, $(x, y) \in P$ if and only if $u(x) < v(y)$.

Let P be a binary relation on a set X and define $R = \{(x, y) \mid (y, x) \notin P\}$. Then P is called *strongly separable* if there exists a countable subset A of X such that

$(x, y) \in P$ implies $\exists x', y' \in A$ such that $(x, x') \in P$, $(x', y') \in R$ and $(y', y) \in P$.

Further Equivalence Results

Gerasimou 2013

Chateauneuf 1987

Theorem (1')

Let X be a topological space and R denote a binary relation on it with symmetric part I and asymmetric part P. Then the following are equivalent.

(a) *X is connected.*

(b) *Every R, R' that are antisymmetric, complete and transitive with closed sections, are either identical or inverse to each other.*

(c) *Every R that is incomplete and transitive with closed sections, and whose P is nontrivial, is fragile.*

(d) *Every R that is asymmetric and has a continuous representation, is strongly separable.*

Sketch of the proof of Theorem 1

Assume (**a**).

(**e**) is due to Schmeidler (Theorem, 1971)

(**c**) follows from (**e**) since Theorem 4 (f) implies that R is transitive.

(**d**) follows from (**c**) since Theorem 4 (g) implies R is semitransitive and Theorem 4 (f) implies R is transitive.

Thm4

(**b**) follows from (**c**) and the observation that any antisymmetric binary relation is semitransitive and its symmetric part is transitive.

Converse:

Assume X is disconnected. Then there exists a nonempty open set $Y \subsetneq X$ which has an open complement Y^c.

Define $R = Y \times Y^c$. Then $P = R$. It is easy to check that R and P satisfy the assumptions of $(\mathbf{b}), (\mathbf{c}), (\mathbf{d}), (\mathbf{e})$.

Since Y and Y^c are nonempty, therefore R is not complete.

Back

Sketch of the proof of Theorem 2

Assume (**a**), i.e. X is 2-connected. If X has only one component, then (b) follows from Theorem 1 (b). Assume X has two components C_1, C_2.

(**b**) Let P denote the asymmetric part of R.

Since R is a complete with closed sections, P has open sections.

Claim. $R^{-1}(x) \cap C_i$ is connected for all $x \in X$ and $i = 1, 2$. *Proof.* Assume $R^{-1}(x) \cap C_i$ is disconnected. Then there exist Y, Y^c nonempty and open subsets of the subspace $R^{-1}(x) \cap C_i$. Since $P(x)$ and $R^{-1}(x)$ are disjoint and covers X, therefore $\{Y, [Y^c \cup (P(y) \cap C_i)]\}$ form an open partition of C_i, hence C_i is disconnected. This furnishes us a contradiction.

Pick $x, y, z \in X$ such that $y \in R(x)$ and $z \in R(y)$. If $x = y$ or $y = z$, then the proof is trivial. For $x \neq y \neq z$, the definition of P implies $y \in P(x)$ and $z \in P(y)$.

Assume $x \notin P^{-1}(z)$. Since $z \neq x$, R is complete and antisymmetric, therefore $z \in P^{-1}(x)$. Since $z \in P(y)$, therefore $X \backslash P(y) \subset X \backslash \{z\}$. Since $y \in P(x)$, therefore $X \backslash P(x) \subset X \backslash \{y\}$. Since $x \in P(z)$, therefore $X \backslash P(z) \subset X \backslash \{x\}$.

Since R is complete and antisymmetric, therefore

$$R^{-1}(y) \subset P^{-1}(z) \cup P(z), \;\; R^{-1}(x) \subset P^{-1}(y) \cup P(y), \;\; R^{-1}(z) \subset P^{-1}(x) \cup P(x).$$

Since C_1, C_2 are components of X, each of x, y, z are contained in one and only one of the components. The following three cases cover all possibilities: (i) $x, y \in C_i$, (ii) $x, z \in C_i, y \in C_j$ and (iii) $x \in C_i, y, z \in C_j$ where $i = 1, 2, i \neq j$.

If $x, y \in C_i$, then Claim implies $R^{-1}(y) \cap C_i$ is connected. Note that $x, y \in R^{-1}(y) \cap C_i$. Moreover, $x \in P(z)$ and $y \in P^{-1}(z)$. Hence, $\{P^{-1}(z) \cap C_i, P(z) \cap C_i\}$ is an open cover of $R^{-1}(y) \cap C_i$. This furnishes us a contradiction. If $x, z \in C_i, y \in C_j$, then Claim implies $R^{-1}(x) \cap C_i$ is connected. Note that $x, z \in R^{-1}(x) \cap C_i$. Moreover, $z \in P(y)$ and $x \in P^{-1}(y)$. Hence, $\{P^{-1}(x) \cap C_i, P(y) \cap C_i\}$ is an open cover of $R^{-1}(x) \cap C_i$. This furnishes us a contradiction. If $x \in C_i, y, z \in C_j$, then Claim implies $R^{-1}(z) \cap C_j$ is connected. Note that $y, z \in R^{-1}(z) \cap C_j$. Moreover, $y \in P(x)$ and $z \in P^{-1}(x)$. Hence, $\{P^{-1}(z) \cap C_j, P(z) \cap C_j\}$ is an open cover of $R^{-1}(z) \cap C_j$. This furnishes us a contradiction.

Therefore, $x \in P^{-1}(z)$, hence R is transitive.

Back

($\mathbf{c}$) Let I denote the symmetric part of R and P denote its asymmetric part. Since R is complete and I is transitive, therefore, I is an equivalence relation. Define a relation $\hat{R}$ on the quotient space $X|I$ with respect to I as $([x],[y]) \in \hat{R}$ if $(x',y') \in R$ for all $x' \in [x]$ and $y' \in [y]$. Define $\hat{P}$ as the asymmetric part of $\hat{R}$. It follows from X is 2-connected that $X|I$ is 2-connected. If $X|I$ has one component, then P is 2-connected implies $\hat{P}$ is nontrivial. Hence, Theorem 1 **a** $\Rightarrow$ **c** implies $\hat{R}$ is transitive. If $X|I$ has two components, then it follows from P is 2-connected that $\hat{P}$ is 2-connected. Hence, **a** $\Rightarrow$ **e** above implies $\hat{R}$ is transitive. Therefore, it follows from the construction of $\hat{R}$ that R is transitive.

($\mathbf{d}$) Theorem 3 implies that R is complete. It follows from ($\mathbf{b}$) that R is also transitive.

($\mathbf{e}$), ($\mathbf{f}$) Theorem 3 implies that R is complete. It follows from ($\mathbf{c}$) that R is also transitive.

(d), **(e)**, **(f)** $\Rightarrow$ **(a)** Assume X has at least three components. Then, as illustrated in the argument in **b** $\Rightarrow$ **a**, there exists a partition $\{Y_1, Y_2, Y_3\}$ of X which is both open and closed. Define a binary relation on X as $R = (Y_1 \times Y_2) \cup (Y_1 \times Y_3) \cup (Y_2 \times Y_3)$. Then, its symmetric part is $I = \emptyset$ and its asymmetric part is $P = R$. By construction, the sections of R is closed and the sections of P are open. Moreover, R is semitransitive and antisymmetric, and I is transitive. Defining $\mathcal{C}^1 = \{Y_1, Y_2\}$ and $\mathcal{C}^2 = \{Y_2, Y_3\}$ implies P is 2-nontrivial. Finally, it is clear that R is incomplete.

(b), **(c)** $\Rightarrow$ **a** The construction is illustrated in the example following Theorem 3.

Back

Sketch of the proof of Theorem 3

(**a**) $\Rightarrow$ (**c**) Assume X is k-connected and $\{C_1, \ldots, C_k\}$ denote the set of components of X. Define $K = \{1, \ldots, k\}$.

Claim 1. Let $x_i \in C_i, x_j \in C_j$. If $(x_i, x_j) \in P$, then $P(x_i) \cup P^{-1}(x_j)$ is both open and closed and contains $C_i \cup C_j$.

Assume there exists $x, y \in X$ such that $(x, y) \notin R \cup R^{-1}$. Then, there exists $i, j \in K$ such that $x \in C_i$ and $y \in C_j$. Since P is k-nontrivial, there exists $x_i \in C_i, x_j \in C_j$ such that $(x_i, x_j) \in P \cup P^*$. Without loss of generality, assume $(x_i, x_j) \in P$. Then, it follows from Claim 1 that $x \in P(x_i) \cup P^{-1}(x_j)$.

Claim 2. If $x \in P^{-1}(x_j)$, then $y \in P^{-1}(x_j)$. If $x \in P(x_i)$, then $y \in P(x_i)$.

It follows from Claim 2 that $x_i \in P^{-1}(x) \cap P^{-1}(y)$ or $x_j \in P(x) \cap P(y)$. Therefore, $\left[P^{-1}(x) \cap P^{-1}(y)\right] \cap C_i \neq \emptyset$ or $[P(x) \cap P(y)] \cap C_j \neq \emptyset$. Since $x \in C_i, y \in C_j$ and $x, y \notin P^{-1}(x) \cap P^{-1}(y)$, therefore $C_i, C_j \not\subset P^{-1}(x) \cap P^{-1}(y)$.

Claim 3. $P(x) \cap P(y)$ and $P^{-1}(x) \cap P^{-1}(y)$ are both open and closed.

It follows from Claim 3 that $\left\{P^{-1}(x) \cap P^{-1}(y) \cap C_i, \left[P^{-1}(x) \cap P^{-1}(y)\right]^c \cap C_i\right\}$ is an open partition of C_i or $\left\{P(x) \cap P(y) \cap C_j, [P(x) \cap P(y)]^c \cap C_j\right\}$ is an open partition of C_j. This furnishes us a contradiction with C_i and C_j being components of X. Therefore, R is complete.

Parts $(\mathbf{b}), (\mathbf{d})$ follows from Theorem 4 (g).

$(\mathbf{c}), (\mathbf{b}), (\mathbf{d}) \Rightarrow$ **a** Assume X has at least three components. Then, as illustrated in the argument in Theorem 2, $(b) \Rightarrow (a)$, there exists a partition $\{Y_1, \ldots, Y_{k+1}\}$ of X which is both open and closed. Define a binary relation on X as

$$R = \bigcup_{i=1}^{k} \left(\bigcup_{j=2}^{k+1} Y_i \times Y_j \right).$$

Then, its symmetric part is $I = \emptyset$ and its asymmetric part is $P = R$. By construction, the sections of R is closed and the sections of P are open. Moreover, R is semitransitive and antisymmetric, and I is transitive. Defining $\mathcal{C}^1 = \{Y_1, \ldots, Y_k\}$ and $\mathcal{C}^2 = \{Y_2, \ldots, Y_{k+1}\}$ implies P is k-nontrivial. Finally, it is clear that R is incomplete.

Back

Sketch of the proof of Theorem 4

(b) Let $X = \{1, 2, 3\}$, $R = \{(1, 2)\}$. It is clear that R is transitive and $P = R$. It follows from $(1, 3) \notin P, (3, 2) \notin P$ and $(1, 2) \in P$ that NT is not satisfied. Now define a relation $R' = \{(1, 2), (2, 1)\}$. Then, $P = \emptyset$, hence NT holds. Since $(1, 1), (2, 2) \notin R$, therefore T is not satisfied.

(d) Assume $y \in P(x)$ and $z \in P(y)$. It follows from $y \in P(x)$ and NT that either $z \in P(x)$ or $y \in P(z)$. Since $z \in P(y)$, therefore $z \in P(x)$, hence PP holds. Now, assume $y \in P(x), z \in I(y)$ and $z \notin P(x)$. It follows from $z \in I(y)$ that $y \notin P(z)$. Then NT implies $y \notin P(x)$. This furnishes us a contradiction. Hence, PI holds. An analogous argument implies IP.

(e) Assume $y \in R(x)$ and $z \in R(y)$. First, recall that d implies PP, PI, IP. If $y \in R^{-1}(x)$ and $z \in R^{-1}(y)$, then II implies $z \in I(x)$, hence $z \in R(x)$. If $y \notin R^{-1}(x)$ or $z \notin R^{-1}(y)$, then it follows from PP, PI, IP that $z \in P(x)$, hence $z \in R(x)$.

(f) Note that P is negatively transitive if and only if $(x, y) \in P$ implies either $(x, z) \in P$ of $(z, y) \in P$ for all $x, y, z \in X$. Pick $x, y \in X$ such that $(x, y) \in P$. Now we will show that $P(x) \cup P^{-1}(y) = R(x) \cup R^{-1}(y)$. It is clear that $P(x) \cup P^{-1}(y) \subset R(x) \cup R^{-1}(y)$. In order to show the converse inclusion, pick $z \in R(x)$. Assume $z \notin P(x) \cup P^{-1}(y)$, i.e. $z \notin P(x)$ and $y \notin P(z)$. It follows from $z \notin P(x)$ and $z \in R(x)$ that $x \in R(z)$. Hence $(z, x) \in I$. It follows from IP and $(z, x) \in I, (x, y) \in P$ that $y \in P(z)$. This furnishes us a contradiction. Now pick $z \in R^{-1}(y)$. Assume $z \notin P(x) \cup P^{-1}(y)$, i.e. $z \notin P(x)$ and $y \notin P(z)$. It follows from $y \notin P(z)$ and $y \in R(z)$ that $z \in R(y)$. Hence $(y, z) \in I$. It follows from PI and $(y, z) \in I, (x, y) \in P$ that $z \in P(x)$. This furnishes us a contradiction. Hence, $P(x) \cup P^{-1}(y) = R(x) \cup R^{-1}(y)$. Since the left side of the equality is an open set and the right side is closed, and X is connected, $P(x) \cup P^{-1}(y) = X$. Therefore P is negatively transitive.

Thm1

(g) Pick $x, y, z \in X$ such that $y \in P(x)$ and $z \in I(y)$. Assume $z \notin P(x)$. Then, one and only one of the following holds: (a) $z \in I(x)$, (b) $x \in P(z)$, (c) $z \in (R(x))^c \cap \left(R^{-1}(x)\right)^c$. If $z \in I(x)$, then *II* implies $y \in I(x)$. This furnishes us a contradiction. Then, it follows from *II* that $I(x) \cap I(z) = \emptyset$. Since $X = I(x) \cup P(x) \cup P^{-1}(x) \cup \left[(R(x))^c \cap \left(R^{-1}(x)\right)^c\right]$, therefore

$$I(z) = [P(x) \cap I(z)] \cup \left[P^{-1}(x) \cap I(z)\right] \cup \left[(R(x))^c \cap \left(R^{-1}(x)\right)^c \cap I(z)\right].$$

It is clear that the three sets in square brackets are pairwise disjoint. Since P has open sections and R has closed sections, the three sets in square brackets are open in $I(z)$. If $x \in P(z)$, then $P(x) \cap I(z)$ and $P^{-1}(x) \cap I(z)$ are nonempty. Then $P(x) \cap I(z)$ and the union of the remaining two sets in square brackets form an open partition of $I(z)$ which contradicts the connectedness of $I(z)$. Analogously, $z \in (R(x))^c \cap \left(R^{-1}(x)\right)^c$ furnishes us a contradiction with the connectedness of $I(z)$. Therefore, $z \in P(x)$, and hence *PI* holds. An analogous argument implies *IP* holds.

Back