Convergence theorems of nonlinear integral functionals

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Definition (nonadditive measure)

Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu\colon \mathcal{A} \to [0, \infty]$ is called a nonadditive measure if it satisfies:

1. $\mu(\emptyset) = 0$ (boundedness from below)
2. $A, B \in \mathcal{A}$, $A \subset B \Rightarrow \mu(A) \le \mu(B)$ (monotonicity)

Nonadditive measures are widely used in theory as well as their applications and have already appeared in many papers: Hausdorff dimension (Hausdorff 1918), lower/upper numerical probability (Koopman 1940), Maharam's submeasure problem (Maharam 1947), capacity (Choquet 1953/54), semivariation (Dunford-Schwartz 1955), quasimeasure (Alexiuk 1968), maxitive measure (Shilkret 1971), participation measure (Tsichritzis 1971), submeasure (Drewnowski 1972, Dobrakov 1974), fuzzy measure (Sugeno 1974), k-triangular set function (Agafanova-Klimkin 1974), game of characteristic function form, distorted measure (Aumann-Shapley 1974), belief/plausibility function (Shafer 1976), possibility measure (Zadeh 1978), pre-measure (Šipoš 1979), necessity measure (Dubois-Prade 1980), approximately additive (Kalton-Roberts 1983), decomposable measure (Weber 1984), Minkowski-Bouligrand dimension (Schroeder 1991), subjective probabilities in decision making, $\cdots\cdots$

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Notation

- $(X, \mathcal{A})$: a measurable space
- $\mathcal{M} := \{\mu \mid \mu\colon \mathcal{A} \to [0, \infty]$ is a nonadditive measure$\}$
- $\mathcal{M}_b := \{\mu \in \mathcal{M} \mid \mu(X) < \infty\}$
- $\mathcal{F} := \{f \mid f\colon X \to [-\infty, \infty]$ is a $\mathcal{A}$-measurable function$\}$
- $\mathcal{F}^+ := \{f \in \mathcal{F} \mid f \geq 0\}$
- The μ-essential boundedness constant

$$\|f\|_\mu := \inf\{r > 0 \mid \mu(\{f \geq r\}) = 0 \text{ and } \mu(\{f \geq -r\}) = \mu(X)\}, \quad f \in \mathcal{F},$$

which is the usual μ-essential supremum for $f \in \mathcal{F}^+$.

Nonlinear integrals

INPUT	AGGREGATION	OUTPUT
(μ, f)	$\longmapsto$	$I(\mu, f)$

- Convergence theorem = Continuity of aggregation process
- Guarantee for certain robustness, consistency, and a non chaotic behavior

Nonlinear integrals appear when aggregating quantities supplied by a measurable function f through a nonadditive measure μ. We consider four types of nonlinear integrals. They are typical and widely used in theory and its applications, such as

- subjective evaluation
- decision-making
- expected-utility theory
- economic model under Knightian uncertainty
- data mining

among others.

Definition (Choquet and Šipoš)

Let $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$.

- Choquet integral: $\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f \geq t\})dt$
- Šipoš integral: $\mathrm{Si}(\mu, f) := \lim_{P \in \Delta^+} \sum_{i=1}^{n} (a_i - a_{i-1})\,\mu(\{f \geq a_i\}),$

  where Δ^+ is the directed set with usual set inclusion of all partitions of $[0, \infty]$ of the form $P = \{a_1, \ldots, a_n\}$ with $0 = a_0 < a_1 < \cdots < a_n < \infty$.

- G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953–54) 131–295.
- J. Šipoš, Integral with respect to a pre-measure, Math. Slovaca 29 (1979) 141–155.

Proposition (Honest extension of the Lebesgue integral)

1. $\mathrm{Ch}(\mu, f) = \mathrm{Si}(\mu, f) = \mathrm{Le}(\mu, f)$ *if* μ *is* σ*-additive.*
2. $\mathrm{Ch}(\mu, f) = \mathrm{Si}(\mu, f)$ *for any* $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$.

Although the Šipoš and the Choquet integral are equal, the Šipoš integral remains important, because it can be defined and developed without any essential knowledge of the Lebesgue integral!

Besides, we have another type of nonlinear integrals.

- The Sugeno integral is defined by lattice operations "$\vee :=$ sup" and "$\wedge :=$ inf" that are important in fuzzy theory originated by Zadeh in 1965.
- By contrast, the Shilkret integral is defined by "supremum" and the usual "multiplication."

Definition (Sugeno and Shilkret)

Let $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$.

- Sugeno integral:

$$\mathrm{Su}(\mu, f) := \sup_{t \in [0,\infty]} [t \wedge \mu(\{f \geq t\})]$$

- Shilkret integral:

$$\mathrm{Sh}(\mu, f) := \sup_{t \in [0,\infty]} [t \cdot \mu(\{f \geq t\})]$$

- M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Dissertation, Tokyo Institute of Technology, Tokyo, 1974.
- D. Ralescu, G. Adams, The fuzzy integral, J. Math. Anal. Appl. 75 (1980) 562–570.
- N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971) 109–116.
- Ru-Huai Zhao, (N) fuzzy integral (in Chinese), J. Math. Res. Exposition 2 (1981) 55–72.

Some of known convergence theorems for nonlinear integrals

Let $\mu \colon \mathcal{A} \to [0, \infty]$ be a nonadditive measure on a measurable space $(X, \mathcal{A})$. Let $f, f_n \colon X \to [0, \infty]$ $(n = 1, 2, \dots)$ be $\mathcal{A}$-measurable functions. Among others the MCT and the BCT are very important and in fact yield other convergence theorems such as the FL and the DCT.

- The monotone increasing convergence theorem (MICT): Assume that μ is continuous from below and $f_n \uparrow f$ pointwise.
  - ⋆ $I(\mu, f_n) \to I(\mu, f)$ for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$.

    Wang1997 for Ch, Šipoš1979 for Si, Ralescu-Adams1980 for Su, Zhao1981 for Sh
- The monotone decreasing convergence theorem (MDCT): Assume that μ is continuous from above and $f_n \downarrow f$ pointwise.
  - ⋆ $I(\mu, f_n) \to I(\mu, f)$ for $I = \mathrm{Ch}, \mathrm{Si}$ if $I(\mu, f_1) < \infty$. Wang1997 for Ch, Šipoš1979 for Si
  - ⋆ $\mathrm{Su}(\mu, f_n) \to \mathrm{Su}(\mu, f)$ if $\mu(\{f_1 > \mathrm{Su}(\mu, f)\}) < \infty$. Wang1984
  - ⋆ $\mathrm{Sh}(\mu, f_n) \to \mathrm{Sh}(\mu, f)$ if $\mu(X) < \infty$ and f_1 is μ-essentially bounded. Zhao1981
- The bounded convergence theorem (BCT): Assume that μ is autocontinuous, $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially bounded and $f_n \to f$ in μ-measure.
  - ⋆ $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$. Murofushi *et al.*1997

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The purpose of the talk

Propose a unified approach to those convergence theorems for nonlinear integrals and give "one-size-fits-all" results!

1. Regard nonlinear integrals as a nonlinear functional $I\colon \mathcal{M}\times\mathcal{F}^+ \to [0,\infty]$, where $\mathcal{M}$ is the set of all nonadditive measures on $(X,\mathcal{A})$ and $\mathcal{F}^+$ is the set of all $\mathcal{A}$-measurable functions $f\colon X\to[0,\infty]$.
2. The key concept is a perturbation of functional I, that is,

   there are two families of control functions $\{\varphi_p\}_{p>0}, \{\psi_q\}_{q>0} \subset \Phi$ satisfying the following perturbation: for any $\mu\in\mathcal{M}$, $f,g\in\mathcal{F}^+$, $\varepsilon\geq 0$, $\delta\geq 0$, $p>0$, and $q>0$, it holds that

   $$I(\mu,f)\leq I(\mu,g)+\varphi_p(\delta)+\psi_q(\varepsilon)$$

   if $\|f\|_\mu < p$, $\mu(X)<q$, and $\mu(\{f\geq t\})\leq\mu(\{g+\varepsilon\geq t\})+\delta$ for all $t\in\mathbb{R}$, where Φ is the set of all functions $\varphi\colon[0,\infty)\to[0,\infty)$ satisfying $\varphi(0)=\lim_{t\to+0}\varphi(t)=0$.

   This perturbation manages not only the monotonicity of functional but also the small change of the functional value arising as a result of adding small amounts to the function f and its μ-distribution function $\mu(\{f\geq t\})$.

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To discuss convergence theorems regardless of the type of nonlinear integrals and develop a unifying approach to formulation and proof, we introduce some classes of nonlinear functionals.

- $I\colon \mathcal{M} \times \mathcal{F}^+ \to [0,\infty]$ is an *integral*, i.e.,
  1. $I(\mu, 0) = 0$ for every $\mu \in \mathcal{M}$
  2. $I(\mu, f) \le I(\mu, g)$ for every $\mu \in \mathcal{M}$ and $f, g \in \mathcal{F}^+$ with $f \le g$
- The functional value $I(\mu, f)$ is often called the μ-integral of f.

Definition (generative)

I is *generative* $\overset{def}{\iff}$ there is $\theta\colon [0,\infty]^2 \to [0,\infty]$ such that

$$I(\mu, r\chi_A) = \theta(r, \mu(A))$$

for every $\mu \in \mathcal{M}$, $r \in [0,\infty]$, and $A \in \mathcal{A}$. The function θ is called a *generator* of I.

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Among others, $\theta(a, b) := a \cdot b$ and $\theta(a, b) := a \wedge b$ are typical examples of generators of integral functionals satisfying the following useful properties:

Definition (limit preserving generator of finite and continuous type)

- θ is *limit preserving* $\overset{def}{\iff}$ for any $\{b_n\}_{n\in\mathbb{N}} \subset [0, \infty]$ and $b \in [0, \infty]$, it holds that $b_n \to b$ whenever $\theta(r, b_n) \to \theta(r, b)$ for every $r \in (0, \infty)$.
- θ is *of finite type* $\overset{def}{\iff}$ $\theta(a, b) < \infty$ whenever $a, b \in [0, \infty)$.
- θ is *of continuous type* $\overset{def}{\iff}$ θ is continuous on $D := [0, \infty]^2 \setminus \{(0, \infty), (\infty, 0)\}$.

Definition (elementary)

I is *elementary* $\overset{def}{\iff}$ I is generative with generator θ and there is a pseudo-addition $\oplus\colon [0,\infty]^2 \to [0,\infty]$ such that

$$I\left(\mu, \bigoplus_{i=1}^{n} (r_i \ominus r_{i-1})\chi_{A_i}\right) = \bigoplus_{i=1}^{n} \theta\big(r_i \ominus r_{i-1}, \mu(A_i)\big)$$

for any $\mu \in \mathcal{M}$, $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{A}$, and $r_1, \ldots, r_n \in (0,\infty)$ with $A_1 \supset \cdots \supset A_n$ and $0 = r_0 < r_1 < \cdots < r_n$.

- Pseudo-addition $\oplus\colon [0,\infty]^2 \to [0,\infty]$ is a binary operation that is commutative, associative, non-decreasing in both components, continuous, and 0 is its neutral element.
- Pseudo-difference $\ominus\colon [0,\infty]^2 \to [0,\infty]$ is defined by

$$a \ominus b := \inf\{x \in [0,\infty] \colon b \oplus x \geq a\}$$

  for each $a, b \in [0,\infty]$.
- $a \ominus b = a - b$ if $a \oplus b = a + b$ and $a > b$.
- $a \ominus b = a$ if $a \oplus b = a \vee b$ and $a > b$.

Proposition (elementariness of Ch, Si, Su, and Sh)

The integral functionals Le, Ch, Si, Su, Sh *are all generative and elementary with limit preserving generators*

$$\theta(a,b) := a \cdot b,\ a \cdot b,\ a \cdot b,\ a \wedge b,\ a \cdot b$$

of finite and continuous type with respect to the pseudo-addition $+$, $+$, $+$, $\vee$, $\vee$, *respectively.*

- In this talk, a perturbation of integral functional is a key concept for formulating “one-size-fits-all” convergence theorems for nonlinear integrals.
- To formulate a perturbation of functional, we need the following notion of the dominance of pairs of a set function and a function.

Definition (dominance of pairs of a set function and a function)

Let $\mu, \nu \colon \mathcal{A} \to [0, \infty]$ be set functions and $f, g \in \mathcal{F}$.

- $(\mu, f) \prec (\nu, g) \overset{def}{\iff} \mu(\{f \geq t\}) \leq \nu(\{g \geq t\})$ for every $t \in \mathbb{R}$.

Then (μ, f) is called *dominated* by (ν, g).

In what follows, let

$$\Phi := \left\{ \varphi \,\middle|\, \varphi \colon [0, \infty) \to [0, \infty),\ \varphi(0) = \lim_{t \to +0} \varphi(t) = 0 \right\}$$

be the set of control functions.

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Recall that

$$\Phi := \left\{ \varphi \,\middle|\, \varphi\colon [0,\infty) \to [0,\infty),\ \varphi(0) = \lim_{t\to+0} \varphi(t) = 0 \right\}$$

is the set of all control functions.

Definition (perturbation)

Let $I\colon \mathcal{M} \times \mathcal{F}^+ \to [0,\infty]$ be a functional.

- I is *perturbative* $\overset{def}{\Longleftrightarrow}$ there are two families of control functions $\{\varphi_p\}_{p>0}$, $\{\psi_q\}_{q>0} \subset \Phi$ satisfying the following *perturbation*: for any $\mu \in \mathcal{M}$, $f, g \in \mathcal{F}^+$, $\varepsilon \geq 0$, $\delta \geq 0$, $p > 0$, and $q > 0$, it holds that

$$I(\mu, f) \leq I(\mu, g) + \varphi_p(\delta) + \psi_q(\varepsilon)$$

whenever $\|f\|_\mu < p$, $\mu(X) < q$, and $(\mu, f) \prec (\mu + \delta, g + \varepsilon)$.

This perturbation manages not only the monotonicity of I but also the small change of the functional value $I(\mu, f)$ arising as a result of adding small amounts ε and δ to the function f and its μ-decreasing distribution function $\mu\{f \geq t\}$).

We also need the following supplementary classes of functionals.

Definition (upper marginal continuous, measure-truncated)

Let $I\colon \mathcal{M}\times\mathcal{F}^+ \to [0,\infty]$ be a functional.

- I is *upper marginal continuous* $\overset{def}{\Longleftrightarrow}$ for every $(\mu, f) \in \mathcal{M}\times\mathcal{F}^+$, it holds that

  $I(\mu, f) = \sup_{r>0} I(\mu, f\wedge r)$.
- I is measure-truncated $\overset{def}{\Longleftrightarrow}$ for every $(\mu, f) \in \mathcal{M}\times\mathcal{F}^+$, it holds that

  $I(\mu, f) = \sup_{s>0} I(\mu\wedge s, f)$.

Proposition (perturbation of Ch, Si, Su, and Sh)

The integral functionals Le, Ch, Si, Su, Sh *are all perturbative with control functions* $\varphi_{p,q}(t) := pt, pt, pt, p\wedge t, pt$ *and* $\psi_{p,q}(t) := qt, qt, qt, q\wedge t, qt$*, respectively. Moreover, they are all upper marginal continuous and measure-truncated.*

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Limit theorems for integral functionals

Using the perturbation of integral functional, together with a perturbative method of proof, we obtain "one-size-fits-all" convergence theorems for nonlinear integrals. As to the MICT, we have

Theorem (MICT: The Monotone Increasing Convergence Theorem)

Let $I\colon \mathcal{M}\times\mathcal{F}^+ \to [0,\infty]$ be an integral functional. Let $\mu\in\mathcal{M}$. Consider the following two assertions:

1. *μ is continuous from below.*
2. *The MICT holds for $I(\mu,\cdot)$, that is, $I(\mu,f_n)\to I(\mu,f)$ for every increasing $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{F}^+$ with pointwise limit $f\in\mathcal{F}^+$.*

(1) *If I is upper marginal continuous, measure-truncated, perturbative, and elementary with generator of continuous type, then ❶ implies ❷.*

(2) *If I is generative with limit preserving generator, then ❷ implies ❶.*

Since $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$ are upper marginal continuous, measure-truncated, perturbative, and elementary with generator of continuous type, as an immediate corollary to our "one-size-fits-all" MICT we have the following MICT for the integrals Ch, Si, Su, and Sh.

Corollary (MICT for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$)

Let $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$. If $\mu \in \mathcal{M}$ is continuous from below and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ is an increasing sequence with pointwise limit $f \in \mathcal{F}^+$, then $I(\mu, f_n) \to I(\mu, f)$.

Next we turn to the MDCT that is NOT an easy consequence of the MICT for the lack of linearity of nonlinear integrals!

Since $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$ are upper marginal continuous, measure-truncated, perturbative, and elementary with generator of continuous type, as an immediate corollary to our "one-size-fits-all" MICT we have the following MICT for the integrals Ch, Si, Su, and Sh.

Corollary (MICT for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$)

Let $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$. If $\mu \in \mathcal{M}$ is continuous from below and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ is an increasing sequence with pointwise limit $f \in \mathcal{F}^+$, then $I(\mu, f_n) \to I(\mu, f)$.

Next we turn to the MDCT that is NOT an easy consequence of the MICT for the lack of linearity of nonlinear integrals!

As to the monotone decreasing convergence theorem, we need some finiteness assumptions on the functional I, the measure μ, and the function f_1.

Theorem (MDCT: The Monotone Decreasing Convergence Theorem)

Let $I\colon \mathcal{M}\times\mathcal{F}^+ \to [0,\infty]$ be an integral functional. Let $\mu \in \mathcal{M}$. Consider the following two assertions:

1. *μ is continuous from above.*
2. *The MDCT holds for $I(\mu,\cdot)$, that is, $I(\mu, f_n) \to I(\mu, f)$ for every decreasing $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ with pointwise limit f satisfying the following conditions:*
   (i) $I(\mu, f_1) < \infty$
   (ii) *uniformly μ-truncated for I, that is, for any $\varepsilon > 0$, there is $c > 0$ such that*

$$I(\mu, f_n) - \varepsilon \le I(\mu, f_n \wedge c) \text{ for all } n \in \mathbb{N}.$$

(1) *If μ is finite and I is perturbative and elementary with generator of continuous type, then ❶ implies ❷.*

(2) *If I is upper marginal continuous and generative with limit preserving generator of finite type, then ❷ implies ❶.*

Thus we have the following.

Corollary (MDCT for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$)

Let $\mu \in \mathcal{M}$ be continuous from above and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ a decreasing sequence with pointwise limit $f \in \mathcal{F}^+$.

1. *Let $I = \mathrm{Ch}, \mathrm{Si}$. If $I(\mu, f_1) < \infty$, then $I(\mu, f_n) \to I(\mu, f)$.*
2. *If $\mu(\{f_1 > \mathrm{Su}(\mu, f)\}) < \infty$, then $\mathrm{Su}(\mu, f_n) \to \mathrm{Su}(\mu, f)$.*
3. *If $\mu(\{f_1 > 0\}) < \infty$ and f_1 is μ-essentially bounded, then $\mathrm{Sh}(\mu, f_n) \to \mathrm{Sh}(\mu, f)$.*

The finiteness assumptions on I, μ, and f_1 such as

- $I(\mu, f_1) < \infty$ for $I = \mathrm{Ch}, \mathrm{Si}$
- $\mu(\{f_1 > \mathrm{Su}(u, f)\}) < \infty$
- $\mu(\{f_1 > 0\}) < \infty$ and f_1 is μ-essentially bounded

Thus we have the following.

Corollary (MDCT for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$)

Let $\mu \in \mathcal{M}$ be continuous from above and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ a decreasing sequence with pointwise limit $f \in \mathcal{F}^+$.

1. *Let $I = \mathrm{Ch}, \mathrm{Si}$. If $I(\mu, f_1) < \infty$, then $I(\mu, f_n) \to I(\mu, f)$.*
2. *If $\mu(\{f_1 > \mathrm{Su}(\mu, f)\}) < \infty$, then $\mathrm{Su}(\mu, f_n) \to \mathrm{Su}(\mu, f)$.*
3. *If $\mu(\{f_1 > 0\}) < \infty$ and f_1 is μ-essentially bounded, then $\mathrm{Sh}(\mu, f_n) \to \mathrm{Sh}(\mu, f)$.*

The finiteness assumptions on I, μ, and f_1 such as

- $I(\mu, f_1) < \infty$ for $I = \mathrm{Ch}, \mathrm{Si}$
- $\mu(\{f_1 > \mathrm{Su}(u, f)\}) < \infty$
- $\mu(\{f_1 > 0\}) < \infty$ and f_1 is μ-essentially bounded

The finiteness assumptions on I, μ, and f_1 such as

- $I(\mu, f_1) < \infty$ for $I = \mathrm{Ch}, \mathrm{Si}$
- $\mu(\{f_1 > \mathrm{Su}(u, f)\}) < \infty$
- $\mu(\{f_1 > 0\}) < \infty$ and f_1 is μ-essentially bounded

1. assure that every decreasing $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ is uniformly μ-truncated for I, that is, for any $\varepsilon > 0$, there is $c > 0$ such that
$$I(\mu, f_n) - \varepsilon \leq I(\mu, f_n \wedge c) \text{ for all } n \in \mathbb{N}.$$
2. reduce the MDCT, which is in fact valid for even infinite nonadditive measures, to our "one-size-fits-all" MDCT proved for finite ones by defining the corresponding finite nonadditive measures by
   - $\nu(A) := \mu(A \cap \{(f_1 > r_0\})$ for $I = \mathrm{Ch}, \mathrm{Si}$, where $r_0 > 0$ is chosen so that $I(\mu, f_1 \wedge r_0)$ is sufficiently small (It is possible since $I(\mu, f_1) < \infty$!)
   - $\nu(A) := \mu(A \cap \{f_1 > \mathrm{Su}(\mu, f)\})$ for $I = \mathrm{Su}$
   - $\nu(A) := \mu(A \cap \{f_1 > 0\})$ for $I = \mathrm{Si}$

   depending on each integral.

In addition, they cannot be dropped.

As to a "one-size-fits-all" bounded convergence theorem, we have

Theorem (BCT: The Bounded Convergence Theorem)

Let $I\colon \mathcal{M}\times\mathcal{F}^+ \to [0,\infty]$ be an integral functional. Let $\mu\in\mathcal{M}_b$. Consider the following two assertions:

1. *μ is autocontinuous, that is, $\mu(A\triangle B_n)\to\mu(A)$ whenever $A, B_n\in\mathcal{A}$ and $\mu(B_n)\to 0$.*
2. *The BCT holds for $I(\mu,\cdot)$, that is, $I(\mu,f_n)\to I(\mu,f)$ for every uniformly μ-essentially bounded $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{F}^+$ converging in μ-measure to $f\in\mathcal{F}^+$.*

If I is perturbative, then ❶ implies ❷. If I is generative with limit preserving generator, then ❷ implies ❶.

Example (autocontinuous measures)

1. Every subadditive measure λ and its distortion, for example,
$$\mu(A) := \lambda(A)^2 + \sqrt{\lambda(A)}.$$
2. Every nonadditive measure satisfying $\inf\{\mu(A)\colon A\neq\emptyset\} > 0$.

Since $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$ are perturbative, we have the following corollary.

Corollary (BCT for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$)

Let $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$. If $\mu \in \mathcal{M}_b$ is autocontinuous and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+$ is a uniformly μ-essentially bounded sequence converging in μ-measure to $f \in \mathcal{F}^+$, then $I(\mu, f_n) \to I(\mu, f)$.

- If μ is null-additive, that is, $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ and $\mu(B) = 0$, then $I(\mu, f) = I(\mu, g)$ for $I = \mathrm{Ch}, \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$ whenever $f = g$ μ-a.e. We thus have a.e. versions of the MICT and MDCT.
- If μ is strongly order continuous, that is, $\mu(A_n) \to 0$ whenever $A_n, A \in \mathcal{A}$, $A_n \downarrow A$, and $\mu(A) = 0$, then by the Lebesgue theorem $f_n \to f$ μ-a.e. implies $f_n \to f$ in μ-measure. Thus we have a.e. versions of the BCT.
- Let $I\colon \mathcal{M} \times \mathcal{F}^+ \to [0, \infty]$ be an integral functional. The *symmetric extension* of I is defined by

$$I^s(\mu, f) := I(\mu, f^+) - I(\mu, f^-), \quad (\mu, f) \in \mathcal{M} \times \mathcal{F}$$

and the *asymmetric extension* of I is defined by

$$I^a(\mu, f) := I(\mu, f^+) - I(\bar{\mu}, f^-), \quad (\mu, f) \in \mathcal{M}_b \times \mathcal{F},$$

where both extensions are not defined if the right hand side is $\infty - \infty$. Then we can extend our MICT/MDCT and BCT for such extensions.

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Thank you very much for your attention!