Unbounded norm topology

Marko Kandić

Faculty of Mathematics and Physics
University of Ljubljana

marko.kandic@fmf.uni-lj.si

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This new convergence is called the unbounded convergence.

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- *(x_n) is uo-convergent;*
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In this case, (x_n) is order bounded and their limits in (1), (3) in (4) are the same.

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for each $y \in X_+$. Notation: $x_\alpha \xrightarrow{\mathrm{un}} x$.

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- $L_p(\mu)$ $(1 \leq p < \infty)$ with μ finite: un-convergence $=$ convergence in measure.
- $C_0(\Omega)$ where Ω is locally compact Hausdorff: un-convergence $=$ uniform convergence on compact subsets of Ω.

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$x_\alpha \xrightarrow{\text{un}} 0$ iff for each subnet y_β there exists a further subnet z_γ such that $z_\gamma \xrightarrow{\text{un}} 0$.

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It is easy to see

$$x_\alpha \xrightarrow{\text{un}} 0 \qquad \Longleftrightarrow \qquad x_\alpha \xrightarrow{\tau} 0.$$

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un-topology = norm topology iff X is lattice isomorphic to $C(K)$.

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If u is a quasi-interior point, (the) metric is given by

$$d(x,y) = \||x-y| \wedge u\|.$$

Witness of un-convergence

If $u \in X_+$ is a quasi-interior point, then

$$\begin{aligned} x_\alpha \xrightarrow{\text{un}} 0 \qquad &\iff \qquad d(x_\alpha, 0) \to 0 \\ &\iff \qquad \||x_\alpha| \wedge u\| \to 0. \end{aligned}$$

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- If X is order continuous, then X is un-complete iff $\dim X < \infty$.

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- Since X always has weak units, un-topology is metrizable.
- $y_\alpha \xrightarrow{\text{un}-X} 0$ in $L_0(\mu)$ iff $y_\alpha|_A \xrightarrow{\mu} 0$ whenever $\mu(A) < \infty$.

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Let X be an order complete Banach lattice.

- *Un-topology on X^u is Hausdorff.*
- *Un-topology on X^u is metrizable iff X has a quasi-interior point.*

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Theorem (K., Li, Troitsky)

Let X be an order continuous Banach lattice with a weak unit such that X is an order dense ideal in Y. If $y_n \xrightarrow{\text{un}-X} 0$ in Y, then there is a subsequence $y_{n_k} \xrightarrow{\text{uo}} 0$ in Y.

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Thank you for your attention!