Approximation properties

of a Banach space and its subspaces

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1. HAPpy Banach spaces.

2, Joint AP for a Banach space X and its subspace Y.

3. Erdos meets Lidskii.

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Approximation Property

Approximation Property (AP): For any compact set $K \subset X$ there is a sequence of (bounded) linear operators of finite rank on X that converges to the identity uniformly on K.

If the finite rank operators can always be chosen to have norm at most λ, X is said to have the λ-*bounded approximation property* (λ-BAP). Grothendieck called the 1-BAP the metric approximation property.

A reflexive space that has the AP also has the 1-BAP [Grothendieck 1955]. Here the spaces I'll discuss are mostly reflexive or even superreflexive.

HAPpy Banach spaces

WBJ and A. Szankowski, *Hereditary approximation property, Annals Math. (2012).*

A Banach space has the *hereditary approximation property* (HAP) provided every subspace has the approximation property. There are non Hilbertian spaces that have the HAP [J, '80], [Pisier, '88]. All of these examples are *asymptotically Hilbertian*; i.e., for some K and every n, there is a finite codimensional subspace all of whose n-dimensional subspaces are K-isomorphic to ℓ_2^n. An asymptotically Hilbertian space must be superreflexive and cannot have a symmetric basis unless it is isomorphic to a Hilbert space. This led to two problems [J, '80]:

1. Can a non reflexive space have the HAP?

2. Does there exist a non Hilbertian space with a symmetric basis that has the HAP?

The HAP is very difficult to work with. It does not have good permanence properties–there are spaces X and Y that have the HAP s.t. $X \oplus Y$ fails the HAP [Casazza-Garcia-J, '01]. Spaces with the HAP are HAPpy; working with them makes one miserable.

Consequently, one looks for nice properties that imply that a space is HAPpy without implying that the space is isomorphic to a Hilbert space.

Let $D_n(X) := \sup d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X. Classical results Kwapien, Krivine-Maurey-Pisier, Davie-Figiel, Szankowski yield that if X is HAPpy then in some sense X must be close to a Hilbert space; in particular, $D_n(X)$ must grow more slowly than any power of n maybe even more slowly than $\log n$.

2. Does there exist a non Hilbertian space with a symmetric basis that has the HAP?

The main result of [JS '12] gives an affirmative answer to problem 2 from [J, '80]:

Theorem. *There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \le f(n)$, then X has the HAP.*

$D_n(X) := \sup d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X.

There are non Hilbert spaces X with a symmetric basis that satisfy the hypothesis $D_n(X) \le f(n)$ for all n. X can be either a space of Schlumprecht type or an Orlicz sequence space L_Φ with $\Phi(x) = x^2 g(x)$ with $g(x)$ VERY slowly varying.

$D_n(X) :=$ sup $d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X.

Theorem. *There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \le f(n)$, then X has the HAP.*

The proof combines the ideas in [J, '80] with the argument in [Lindenstrauss-Tzafriri, '76] that a uniformly convex space that has the λ-uniform approximation property (UAP) for some λ actually has the 1-UAP. It is basically an averaging argument that turns out to work to not just for sequences of operators whose norms are uniformly bounded but also for operators whose norms grow very slowly provided, of course, that the sequence satisfies all sorts of extra conditions that I do not state here. A variation of the argument in [J, '80] provides the appropriate sequence of finite rank operators. While the approach is simple enough, the argument itself is fairly technical.

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Theorem. *There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \leq f(n)$, then X has the HAP.*

The estimate of $f(n)$ is good enough, by using [Nielsen,Tomczak-Jaegermann '92], to show that $\ell_2(X)$ has the HAP for every weak Hilbert space X that has an unconditional basis, but it remains open whether $\ell_2(X)$ has the HAP for every weak Hilbert space X.

Another application of this theorem is related to the celebrated result of [Lindenstrauss-Tzafriri '71] that if every subspace of X is complemented, then X is isomorphic to a Hilbert space.
T. Oikhberg asked, "If X is a separable Banach space s.t. every subspace of X is isomorphic to a complemented subspace of X, must X be isomorphic to a Hilbert space?"

A Banach space X is *complementably universal* for a class $\mathcal{M}$ of Banach space provided that every space in $\mathcal{M}$ is isomorphic to a complemented subspace of X. [Kadec '71] constructed a separable Banach space with the BAP that is complementably universal for all separable Banach spaces that have the BAP, while [JS '76] proved that there is no separable Banach space that is complementably universal for all separable Banach spaces that have the AP. If a Banach space is complementably universal for all subspaces of itself and has the AP, then it has the HAP, and hence must be "close" to a Hilbert space.

Theorem. *There is a separable, infinite dimensional Banach space not isomorphic to ℓ_2 that is complementably universal for all subspaces of all of its quotients.*

Theorem. *There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \leq f(n)$, then X has the HAP.*

Theorem. *There is a separable, infinite dimensional Banach space not isomorphic to ℓ_2 that is complementably universal for all subspaces of all of its quotients.*

Let X be any non Hilbertian separable Banach space such that $D_{4^n}(X) \leq f(n)$ for all n. Let (E_k) be a sequence of finite dimensional spaces that is dense (in the sense of the Banach-Mazur distance) in the collection of all finite dimensional spaces that are contained in some quotient of $\ell_2(X)$ and let Y be the ℓ_2-sum of the E_k. Then $D_n(Y) \leq f(n)$ for all n. If you are old enough to know the right background, you can give a short argument to prove that Y is complementably universal for all subspaces of all of its quotients.

Joint AP for a Banach space X and its subspace Y

T. Figiel, WBJ, A. Pełczyński, Some approximation properties of Banach spaces and Banach lattices, Israel J. Math. 183 (2011).

Let X be a Banach space, let $Y \subseteq X$ be a closed linear subspace, let $\lambda \geq 1$. The pair (X, Y) is said to have the λ-BAP if for each $\lambda' > \lambda$ and each subspace $F \subseteq X$ with $\dim F < \infty$, there is a finite rank operator $u : X \to X$ such that $||u|| < \lambda'$, $u(x) = x$ for $x \in F$ and $u(Y) \subseteq Y$.

If (X, Y) has the λ-BAP then X/Y has the λ-BAP. Thus by [Szankowski, '09], for $1 \leq p < 2$ there are subspaces Y of ℓ_p that have the BAP and yet (ℓ_p, Y) fails the BAP.

It is open whether (X, Y) has the BAP if X, Y, and X/Y all have the BAP but I don't believe it.

If Y is a finite dimensional subspace of X and X has the λ-BAP then also (X, Y) has the λ-BAP and hence also X/Y has the λ-BAP. That is, the λ-BAP passes to quotients by finite dimensional subspaces. By duality you get that if X^* the λ-BAP then every finite codimensional subspace of X has the λ-BAP. In particular, every finite codimensional subspace of an L_1 space has the 1-BAP. Had anyone previously noticed this?

In fact,

Proposition. *X^* has the λ-BAP iff (X, Y) has the λ-BAP for every finite codimensional subspace Y.*

Proposition. *X^* has the λ-BAP iff (X,Y) has the λ-BAP for every finite dimensional subspace Y.*

Proposition. *Let X be a Banach space and let $Y \subseteq X$ be a closed subspace such that* $\dim X/Y = n < \infty$ *and Y has the $\lambda-BAP$ (resp. Y has the $\lambda-UAP$). Then the pair (X,Y) has the $3\lambda-BAP$ (resp. $3\lambda-UAP$).*

This gives the corollary

Corollary. *If X is a Banach space and Y has the $\lambda-BAP$ for every finite codimensional subspace $Y \subseteq X$, then X^* has the $3\lambda-BAP$.*

Consequently, in contradistinction to the case of commutative L_1 spaces, for every λ there are finite codimensional subspaces Y of the non commutative L_1 space S_1 that fail the λ-BAP because by [Szankowski '81], $B(\ell_2)$ fails the AP.

Corollary. *There is a subspace Y of $\ell_!$ that has the AP but fails the BAP.*

Proof. Start with a subspace X of ℓ_1 that fails the approximation property [Szankowski '78]. From the existence of such a space it follows [WBJ '72] that if we let Z be the ℓ_1-sum of a dense sequence (X_n) of finite dimensional subspaces of X, then Z^* fails the BAP and yet Z has the BAP. Then Y can be the ℓ_1-sum of a suitable sequence of finite codimensional subspaces of Z because of

Corollary. *If X is a Banach space and Y has the $\lambda-BAP$ for every finite codimensional subspace $Y \subseteq X$, then X^* has the $3\lambda-BAP$.*

We will not discuss the main results in [FJP '11] regarding the structure of various classical spaces that are widely used in analysis. Many of them use the following lemma and variations on it.

Lemma. *If X is a $\mathcal{L}_\infty$−space and Y has the BAP then (X,Y) has the BAP and hence also X/Y has the BAP.*

For general spaces X we do not know much about when the implication "X and Y have the BAP implies X/Y has the BAP" holds. Arguably the most interesting question related to this is:

If X is HAPpy (and, say, reflexive, so that AP is equivalent to the 1-BAP), must (X/Y) have the BAP for every subspace Y?

Erdos* meets Lidskii

Research with T. Figiel and A. Szankowski.

Let X be a Banach space and $Y \subseteq X$ a closed linear subspace. The pair (X, Y) is said to have the AP provided that the identity operator on X is the limit, in the topology of uniform convergence on compact sets, of bounded linear finite rank operators on X that map Y into Y.

If the net of finite rank operators can be chosen to be uniformly bounded by λ, say that (X, Y) has the λ-BAP. It turns out (Grothendieck et al) that if X is reflexive and (X, Y) has the AP, then (X, Y) has the 1-BAP.

There is a dual form to the statement that the pair (X, Y) has the AP:

Proposition. (X, Y) *has the AP iff for every nuclear operator* T *on* X *for which* $TX \subset Y$ *and* $T^2 = 0$ *we have* $tr(T) = 0$.

*Truth in advertising: Erdos is John; not Paul.

Proposition. (X, Y) *has the AP iff for every nuclear operator* T *on* X *for which* $TX \subset Y$ *and* $T^2 = 0$ *we have* $tr(T) = 0$*.*

The second condition says that for certain nuclear operators on X, the trace agrees with the spectral trace.

From this you get that (X, Y) has the AP for every subspace Y (we say that X is jointly HAPpy) iff every nuclear operator T on X for which $T^2 = 0$ has zero trace (so that the trace of T equals its spectral trace).

It is open whether a HAPpy space must be jointly HAPpy. However, if $D_n(X) \to \infty$ sufficiently slowly, then X is not just HAPpy but even jointly HAPpy (not written up.)

More generally, we consider the following concept.

Let $\mathcal{N}$ be a nest of closed subspaces of X. The pair $(X, \mathcal{N})$ is said to have the AP provided that the identity operator on X is the limit, in the topology of uniform convergence on compact sets, of bounded linear finite rank operators on X that map Y into Y for every $Y \in \mathcal{N}$.

If the net of finite rank operators can be chosen to be uniformly bounded by λ, say that $(X, \mathcal{N})$ has the λ-BAP.

If X is reflexive and $(X, \mathcal{N})$ has the AP, then $(X, \mathcal{N})$ has the 1-BAP.

$(X, \mathcal{N})$ has the AP provided that the identity operator on X is the limit, in the topology of uniform convergence on compact sets, of bounded linear finite rank operators on X that map Y into Y for every $Y \in \mathcal{N}$.

Theorem. (Erdos meets Lidskii) *TFAE*
1. $(X, \mathcal{N})$ *has the AP for every nest* $\mathcal{N}$*.*
2. every quasi-nilpotent nuclear operator on X *has zero trace.*
3. $tr(T) = \sum \lambda_n(T)$ *for every nuclear operator* T *on* X *whose eigenvalues* $\lambda_n(T)$ *are absolutely summable.*

[Lidskii '59, Grothendieck '56]: Hilbert sp. satisfy (3).

[Erdos '68]: Hilbert spaces satisfy (1).

[Erdos '74]: (1) $\Rightarrow$ (2) for a Hilbert space.

[Pisier '88]: Weak Hilbert spaces satisfy (3).

Consequently, weak Hilbert spaces satisfy (1).

Call X Lidskii if it satisfies the conditions in

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Weak Hilbert spaces are Lidskii [Pisier '88]. Anything else?

To answer this question, we will have to recall one of the many equivalent definitions of a weak Hilbert space.

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in B^n_{X^*}$ and $x = (x_1, \ldots, x_n) \in B^n_X$, let

$$G(\varepsilon, x) = \mathrm{det}[\langle \varepsilon_i, x_j \rangle]^n_{i,j=1}.$$

Define

$$G_n(X) = \sup\{|G(\varepsilon, x)| : \varepsilon \in B^n_{X^*}, x \in B^n_X\}.$$

X is *a weak Hilbert (WH) space* if

$$\sup_n G_n(X)^{\frac{1}{n}} < \infty.$$

Theorem. *If* $\liminf_n G_n(X)^{\frac{1}{n}} < \infty$*, then* X *is Lidskii.*

But are there any non weak Hilbert spaces that satisfy the hypothesis of the theorem?

Are there any non weak Hilbert spaces that satisfy the hypothesis of the theorem?

Theorem. *If* $\liminf_n G_n(X)^{\frac{1}{n}} < \infty$*, then* X *is Lidskii.*

X is *asymptotically Hilbertian of polynomial growth* if there is a constant λ such that there are subspaces $Y_1, Y_2, \dots \subset X$ with $\dim X/Y_n = O(n^\lambda)$ and $\liminf D_n(Y_n) < \infty$. If the Y_n can be chosen to be uniformly complemented, X is *complementably asymptotically Hilbertian of polynomial growth.*

Theorem. *If* X *is complementably asymptotically Hilbertian of polynomial growth then* $\liminf_n G_n(X)^{\frac{1}{n}} < \infty$*.*

In particular, X can be of the form $(\sum \ell_{p_n}^{k_n})_2$ with $p_n \downarrow 2$ and $k_n \uparrow \infty$ fast enough so that X is not isomorphic to a Hilbert space. With these you can see that the direct sum of two Lidskii spaces need not be Lidskii; in fact, need not be HAPpy.

Some Open Problems

Question 1. Suppose $G_n(X)$ is bounded or $G_n(X)^{\frac{1}{n}} \to 1$. Does it imply that X is (isomorphic to) a Hilbert space?

Question 2. If X is isomorphic to a Hilbert space, does it imply that $G_n(X)$ is bounded or that $G_n(X)^{\frac{1}{n}} \to 1$?

Question 3. Is every HAPpy space jointly HAPpy?

Question 4. Is every HAPpy space Lidskii?

Question 5. Suppose that $D_n(X)$ goes to infinity sufficiently slowly. Must X be a Lidskii space?