Positive representations of $C_0(X)$

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Introduction

Representation of $C_0(X, \mathbb{C})$ on Hilbert space

Let X be a locally compact Hausdorff space, $\mathcal{H}$ is a complex Hilbert space. A $*$-homomorphism $\pi\colon C_0(X, \mathbb{C}) \to B(\mathcal{H})$ is given by a spectral measure P:

$$\pi(f) = \int f dP, \ \forall f \in C_0(X, \mathbb{C}),$$

where P takes its values in the orthogonal projections on $\mathcal{H}$.

Question

Is there any similar result for a positive representation of $C_0(X, \mathbb{R})$ on a real Banach lattice E ? That is, for an algebra homomorphism

$$\pi\colon C_0(X, \mathbb{R}) \to \mathcal{L}_r(E),$$

where $\mathcal{L}_r(E)$ is the space of all regular operators on E, and $\pi(C_0(X)_+) \subseteq \mathcal{L}_r(E)_+$.

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If E is a KB-space, then π is given by a spectral measure that takes its values in the positive projections on E; see [1] (M. de Jeu, F. Ruoff).

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If E is a KB-space, then π is given by a spectral measure that takes its values in the positive projections on E; see [1] (M. de Jeu, F. Ruoff).
Everything is in $\mathbb{R}$-setting from now on.

- Riesz representation theorem

- Riesz representation theorem in order context

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Definition: Positive $E-$valued measure (J.D.M. Wright [2])

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$$m(\cup_{n=1}^{\infty}\triangle_n) = \bigvee_{N=1}^{\infty}\sum_{n=1}^{N} m(\triangle_n).$$

Definition: Positive $E-$valued measure (J.D.M. Wright [2])

Ω is an algebra of subsets of X and E is a monotone σ-complete partially ordered vector space. A positive E-valued measure is a set map $m\colon \Omega \to E \cup \{+\infty\}$ such that

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2. $m(\triangle) \in E_+ \cup \{+\infty\}$ for all $\triangle \in \Omega$;
3. σ-additivity: whenever $\{\triangle_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω with $\cup_{n=1}^{\infty}\triangle_n \in \Omega$, then

$$m(\cup_{n=1}^{\infty}\triangle_n) = \bigvee_{N=1}^{\infty}\sum_{n=1}^{N} m(\triangle_n).$$

If $m(X) \in E$, then we say m is finite.

Integration with respect to a positive $E-$valued measure

The spaces of (real valued) functions to work with:

- $\beta(X)$: the set of all Ω-measurable functions on X ;
- $\beta_0(X)$: $f \in \beta(X)$ and $\forall c \in \mathbb{R}$, $m\{\omega\colon |f(\omega)| > c\} \in E$

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- $\mathrm{S}(X)$: $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{\triangle_i}$, where $\{\triangle_i\}_{i=1}^{n}$ is a finite partition of X in Ω and each $\alpha_i \in \mathbb{R}$;
- $\mathrm{S}_0(X)$: $\varphi \in \mathrm{S}(X)$ and $m(supp(\varphi)) \in E$.

For the above sets, we use the pointwise ordering.

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$\mathrm{S}(X)_+$

For $\varphi \in \mathrm{S}(X)_+$ with $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{\triangle_i}$,

$$I_m(\varphi) = \int_X \varphi dm := \sum_{i=1}^{n} \alpha_i m(\triangle_i).$$

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$$I_m(f) = \begin{cases} \bigvee_{n=1}^{\infty} I_m(\varphi_n), & \text{if } f \in \beta_0(X)_+, \\ +\infty, & \text{if } f \in \beta(X)_+ \setminus \beta_0(X)_+, \end{cases} \tag{1}$$

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- The supremum might be $+\infty$.

Integration

$\mathrm{S}(X)_+ \dashrightarrow \beta(X)_+ \dashrightarrow \beta(X)$.

For $f \in \beta(X),\ f = f_+ - f_-$.
If at least one of $I_m(f_+)$, $I_m(f_-)$ is in E, then we define:

$$I_m(f) := I_m(f_+) - I_m(f_-).$$

Borel measures

Let X be a locally compact Hausdorff space, $\mathcal{B}$ the Borel σ-algebra of X, E a monotone complete partially ordered vector space, and $m\colon \mathcal{B} \to E \cup \{+\infty\}$ a positive E-valued measure. Then m is called

- a Borel measure if $m(K) \in E$ for all compact subset K;
- inner regular at $\triangle$ if $m(\triangle) = \bigvee\{m(K)\colon K \text{ is compact and } K \subseteq \triangle\}$;
- outer regular at $\triangle$ if $m(\triangle) = \bigwedge\{m(V)\colon V \text{ is open and } \triangle \subseteq V\}$;

m is called a regular Borel measure if it is a Borel measure, inner regular at all open sets and outer regular at $\mathcal{B}$.

Definition: Normal

Let E be a partially ordered vector space, and $E_n^\sim$ is the order continuous dual of E, E is called normal if $(x, x') \geq 0$ for all $x' \in (E_n^\sim)_+$ implies $x \in E_+$.

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Definition: Monotone order continuous norm

Let $(E, \|.\|)$ be a normed partially ordered vector space, the norm is called:

- monotone σ-order continuous if for any monotone increasing sequence $\{x_n\}_{n=1}^\infty$ in E_+ with $x = \sup_n x_n \in E$, we have $\|x - x_n\| \to 0$.
- monotone order continuous if for any monotone increasing net $\{x_\lambda\}_{\lambda \in \Lambda}$ in E_+ with $x = \sup_\alpha x_\alpha \in E$, we have $\|x - x_\alpha\| \to 0$.

Riesz representation theorem

Let X be a locally compact Hausdorff space, E a monotone complete partially ordered vector space and $\pi\colon \mathrm{C_c}(X) \to E$ is a positive linear map.

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Theorem 1

Let X be a locally compact Hausdorff space, E a monotone complete partially ordered vector space and $\pi\colon \mathrm{C_c}(X) \to E$ is a positive linear map. If E is a normal directed ordered Banach space with a monotone order continuous norm, then there exists unique positive Borel regular E-valued measure m such that $\pi(f) = I_m(f)$ for all $f \in \mathrm{C_c}(X)$.

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$$\{\pi(f)\colon 0 \le f \le 1,\ f \in \mathrm{C_c}(X)\} \text{ is bounded from above in } E,$$

then m is finite.

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Taking $E = \mathbb{R}$, this is original Riesz representation theorem.

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The basic idea is just as for $E = \mathbb{R}$:
For an open subset V, define

$$m(V) := \bigvee \{\pi(f) \colon f \prec V, f \in \mathrm{C_c}(X)\},$$

where $f \prec V$ means $0 \leq f \leq 1$ and $f = 0$ on V^c.

$\{\pi(f)\colon 0 \leq f \leq 1,\ f \in \mathrm{C_c}(X)\}$ is bounded from above in E,

Theorem 2

Let X be a locally compact Hausdorff space, and let E be a monotone complete normal partially ordered space. Suppose $\pi\colon \mathrm{C_c}(X) \to E$ is a positive linear map such that

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The proof is given by applying Theorem 1 to each $\pi_{x'}$, which is defined by $\pi'_x(f) = (\pi(f), x')$, $x' \in (E_n^{\sim})_+$.

Theorem 3

Let X be a locally compact Hausdorff space, and let E be a monotone complete normal partially ordered space. Suppose $\pi\colon \mathrm{C}_0(X) \to E$ is a positive linear map such that

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Theorem 3

Let X be a locally compact Hausdorff space, and let E be a monotone complete normal partially ordered space. Suppose $\pi\colon C_0(X) \to E$ is a positive linear map such that

$$\{\pi(f)\colon 0 \leq f \leq 1,\ f \in C_c(X)\} \text{ is bounded from above in } E,$$

Then there exists a unique finite positive regular Borel E-valued measure m such that $\pi(f) = I_m(f)$ for all $f \in C_0(X)$.

Since m is finite, $C_0(X)$ is Banach lattice and $C_c(X)$ is norm dense in $C_0(X)$.

Positive Representations

Positive representations of $\mathrm{C}_0(X)$

Let X be a locally compact Hausdorff space, $\mathcal{A}$ a monotone complete partially ordered algebra. A positive representation of $\mathrm{C}_0(X)$ on $\mathcal{A}$ is a positive algebra homomorphism

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Considering π as a positive linear map, there is a generating measure $P(= m)$ of π in some suitable cases. Is there any speciality of P?A spectral measure?

In a partially ordered algebra $\mathcal{A}$, the multiplication by a fixed positive element is

(1) monotone σ-order continuous, if for any $b \in \mathcal{A}_+$, $a \in \mathcal{A}$ and an increasing sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ with $a_n \uparrow a$, we have $ba_n \uparrow ba$, $a_n b \uparrow ab$;

(2) monotone order continuous, if then for any $b \in \mathcal{A}_+$, $a \in \mathcal{A}$ and an increasing net $\{a_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$, with $a_\lambda \uparrow a$, we have $ba_\lambda \uparrow ba$, $a_\lambda b \uparrow ab$.

Positive $\mathcal{A}$-valued spectral measure

Let (X, Ω) be a measurable space, $\mathcal{A}$ a monotone σ-complete partially ordered algebra. A positive $\mathcal{A}$-valued measure $P\colon \Omega \to \mathcal{A}_+$ is called a spectral measure, if

$$P(\triangle_1 \cap \triangle_2) = P(\triangle_1)P(\triangle_2),$$

for all $\triangle_1, \triangle_2 \in \Omega$. If $\mathcal{A}$ has an positive algebra unit e, and $P(X) = e$, then we say P is unital.

• In $P(X)$, the multiplication by a fixed positive element must be monotone σ-order continuous.

Positive representation of $\mathrm{C}_0(X)$

Let X be a locally compact Hausdorff space, $\mathcal{A}$ is a monotone complete normal partially ordered algebra, $\pi\colon \mathrm{C}_0(X) \to \mathcal{A}$ is a positive representation.If

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- $\{\pi(f)\colon 0 \leq f \leq 1, f \in \mathrm{C}_c(X)\}$ is bounded from above in $\mathcal{A}$,
- and the multiplication by a fixed positive element on $\mathcal{A}$ is order continuous,

then there exists a unique positive finite Borel regular spectral measure P generating π.

•Theorem 3 (The Riesz representation theorem of $\mathrm{C}_0(X)$) provides a finite measure $P\colon \mathcal{B} \to \mathcal{A}_+$ such that

$$\pi(f) = I_P(f),\ \forall f \in \mathrm{C}_0(X).$$

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$$\pi(f) = I_P(f), \ \forall f \in \mathrm{C}_0(X).$$

• Using π is multiplicative and the uniqueness of the generating measure in riesz representation theorem to show that

$$I_P(gg_0) = I_P(g)I_P(g_0) \ \forall g, g_0 \in \mathrm{B}(X).$$

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Then we have

$$P(A \cap B) = P(A)P(B),\ \forall A, B \in \mathcal{B}.$$

Example

If E is a normal ordered Banach space with a Levi's norm, then any positive representation $\pi\colon \mathrm{C}_0(X) \to \mathcal{L}_{\mathrm{n}}(E)$ is generated by a regular Borel spectral measure.

Levi's norm: every norm bounded increasing(decreasing) net has a supremum(infmum).

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Mehmet Orhon: Banach lattice with order continuous norm.

Example

Let $S_a(\mathcal{H})$ be the self adjoint operators on a complex Hilbert space $\mathcal{H}$.

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Example

Let $S_a(\mathcal{H})$ be the self adjoint operators on a complex Hilbert space $\mathcal{H}$. Any multiplicative linear map $\pi\colon \mathrm{C}_0(X) \to S_a(\mathcal{H})$ is generated by a regular Borel spectral measure P, where P takes values in the orthogonal projections on $\mathcal{H}$.

For a complex valued function $f + ig$, where f, g are real valued functions, we defined the integral as

$$I_P(f + ig) = I_P(f) + iI_P(g).$$

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where P takes its values in the orthogonal projections on $\mathcal{H}$.

Attention ! This is page 2 !

Reference

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Thank you!