Cauchy quotient means and their properties

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Outline

Beta-type functions

Motivated by the relationship between the Euler Gamma function $\Gamma : (0, \infty) \to (0, \infty)$ and the the Beta function $B : (0, \infty)^2 \to (0, \infty)$

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y > 0,$$

we introduce a new class of functions, called beta-type functions.

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Definition [Himmel, Matkowski 2015]

Let $a \geq 0$ be fixed. For $f : (a, \infty) \to (0, \infty)$, the two variable function $B_f : (a, \infty)^2 \to (0, \infty)$ defined by

$$B_f(x, y) = \frac{f(x) f(y)}{f(x+y)}, \qquad x, y > a,$$

is called the *beta-type function*, and f is called its *generator*.

With this definition we have: $B_\Gamma = \mathcal{B}$.

Means and beta-type functions

We are interested in answering when the beta-type function is a bivariable mean. The answer is given in the following

Theorem 1.

Let $f : (0, \infty) \to (0, \infty)$ be an arbitrary function. The following conditions are equivalent:

(i) the beta-type function $B_f : (0, \infty)^2 \to (0, \infty)$ is a bivariable mean, i.e.

$$\min(x, y) \le B_f(x, y) \le \max(x, y), \qquad x, y > 0;$$

(ii) there is an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = 2xe^{\alpha(x)}, \qquad x > 0;$$

(iii) B_f is the harmonic mean in I,

$$B_f(x, y) = \frac{2xy}{x + y}, \qquad x, y > 0.$$

Mean in an interval

Definition 2.

Let $I \subseteq \mathbb{R}$ be a non-empty interval, $k \in \mathbb{N}$, $k \geq 2$, and $M : I^k \to \mathbb{R}$. The function M is called a mean in the interval I, if

$$\min(x_1, \ldots, x_k) \leq M(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k)$$

holds true for all $x_1, \ldots, x_k \in I$.

Beta-type functions as k-variable means

Theorem 3.

Let $k \in \mathbb{N}, k \geq 2$, be fixed, let $f : (0, \infty) \to (0, \infty)$ and $B_{f,k} : (0, \infty)^k \to (0, \infty)$ defined by

$$B_{f,k}(x_1, \ldots, x_k) := \frac{f(x_1) \cdots f(x_k)}{f(x_1 + \cdots + x_k)}, \qquad x_1, \ldots, x_k > 0.$$

The following conditions are equivalent:

(i) $B_{f,k}$ *is a mean in* $(0, \infty)$*;*

(ii) *there is an additive function* $\alpha : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = k \sqrt[k-1]{x} e^{\alpha(x)}, \quad x > 0;$$

(iii) $B_{f,k}$ *is the beta-type mean, i.e.*

$$B_{f,k}(x_1, \ldots x_k) = \sqrt[k-1]{\frac{k x_1 \cdots x_k}{x_1 + \ldots + x_k}}, \qquad x_1, \cdots, x_k > 0.$$

Beta-type mean

Definition 4.

For any $k \in \mathbb{N}$, $k \geq 2$, the function $\mathfrak{B}_k : (0,\infty)^k \to (0,\infty)$ defined by

$$\mathfrak{B}_k(x_1,\ldots,x_k) = \sqrt[k-1]{\frac{kx_1\cdots x_k}{x_1+\ldots+x_k}}, \qquad x_1,\cdots,x_k > 0$$

is called the k-variable *beta-type mean.*

Cauchy quotients

- **beta-type function (exponential Cauchy quotient)**

$$B_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) \cdot \ldots \cdot f(x_k)}{f(x_1 + \ldots + x_k)}$$

- **logarithmic Cauchy quotient**

$$L_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) + \ldots + f(x_k)}{f(x_1 \cdot \ldots \cdot x_k)}$$

- **multiplicative (or power) Cauchy quotient**

$$P_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) \cdot \ldots \cdot f(x_k)}{f(x_1 \cdot \ldots \cdot x_k)}$$

- **additive Cauchy quotient**

$$A_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) + \ldots + f(x_k)}{f(x_1 + \ldots + x_k)}$$

Questions on Cauchy quotients

where $f : I \to (0, \infty)$ is an arbitrary function defined on a suitable interval, and we asked:

- When is $B_{f,k}$ a mean?
- When is $L_{f,k}$ a mean?
- When is $P_{f,k}$ a mean?
- When is $A_{f,k}$ a mean?

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- When is $A_{f,k}$ a mean?
  Answer:
- In each of the first three cases there exists *exactly one mean that can be written in the form of a beta-type function, a logarithmic Cauchy quotient or a power Cauchy quotient, respectively.*
- *No mean of the form $A_{f,k}$ - in any interval I.*

When $L_{f,k}$ is a mean?

Theorem 5.

Let $k \in \mathbb{N}$, $k \geq 2$, be fixed, $f : (1,\infty) \to (0,\infty)$ be an arbitrary function. The following conditions are equivalent:

(i) *the function $L_{f,k} : (1,\infty)^k \to (0,\infty)$ defined by*

$$L_{f,k}(x_1,\dots,x_k) := \frac{\sum_{j=1}^{k} f(x_j)}{f\left(\prod_{j=1}^{k} x_j\right)}, \qquad x_1,\dots,x_k \in (1,\infty),$$

is a mean;

(ii) *there is $c > 0$ such that*

$$f(x) = \frac{c}{x^{\frac{1}{k-1}}} \log x, \qquad x \in (1,\infty);$$

When $L_{f,k}$ is a mean? (2)

Theorem 7 (continuation)

(iii) $L_{f,k}$ is of the form

$$L_{f,k}(x_1, \dots x_k) = \frac{\sum_{i=1}^{k} \sqrt[k-1]{\prod_{j=1, j\neq i}^{k} x_j} \log x_i}{\sum_{i=1}^{k} \log x_i}, \qquad x_1, \dots, x_k \in (1, \infty).$$

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Remark

An analogous result for $L_{f,k}$ holds true on the domain $(0,1)$.

- The above mean for $k=2$ belongs to the class of Beckenbach-Gini means.

When $P_{f,k}$ is a mean?

Theorem 6.

Let $k \in \mathbb{N}$, $k \geq 2$, be fixed and $f : (1, \infty) \to (0, \infty)$ continuous. The following conditions are equivalent:

(i) $P_{f,k} : (1, \infty)^k \to (0, \infty)$ *defined by*

$$P_{f,k}(x_1, \ldots, x_k) = \frac{f(x_1) \cdots f(x_k)}{f(x_1 \cdots x_k)}, \quad x_1, \ldots, x_k > 1.$$

is a translative mean;

(ii) *there exists $b \in \mathbb{R}$ such that*

$$f(x) = x^{-\frac{\log \log x + b}{k \log k}}, \qquad x > 1;$$

When $P_{f,k}$ is a mean? (2)

Theorem 8 (continuation)

(iii) $P_{f,k}$ is of the form

$$P_{f,k}(x_1,\ldots,x_k) = \left(\prod_{j=1}^{k} x_j^{\log \frac{\log(x_1\cdot\ldots\cdot x_k)}{\log x_j}}\right)^{\frac{1}{k\log k}}, \quad x_1,\ldots,x_k > 1.$$

A class of functions without means

Theorem 7.

Let $k \in \mathbb{N}$, $k \geq 2$, and $a > 0$ be fixed. There is no $f : [a, \infty) \to (0, \infty)$ such that the function $A_{f,k} : [a, \infty)^k \to (0, \infty)$ defined by

$$A_{f,k}(x_1, \ldots, x_k) := \frac{\sum_{j=1}^{k} f(x_j)}{f\left(\sum_{j=1}^{k} x_j\right)}, \qquad x_1, \ldots, x_k \geq a,$$

or $\frac{1}{A_{f,k}}$ is a mean.

Summary

- Beta-type functions naturally generalize the Euler Beta function.
- A two-variable beta-type function is a mean if, and only if, it is the harmonic mean.
- Beta-type functions of k-variables give a homogeneous mean, called beta-type mean, which is neither harmonic nor quasi-arithmetic for $k \geq 3$.
- $L_{f,k}$ and $\frac{1}{L_{f,k}}$ exhibit means related to Beckenbach-Gini means.
- There exists a mean in terms of $P_{f,k}$ and $\frac{1}{P_{f,k}}$.
- There does not exist any mean of the form $A_{f,k}$ or $\frac{1}{A_{f,k}}$.

Remark 1

Let $k \in \mathbb{N}$, $k \geq 2$; $a \geq 0$, and I be as in the Definition 1, and let $f, g : I \to (0, \infty)$. The beta-type functions have the following properties:

(i) (equality) $B_{f,k} = B_{g,k}$ iff there is a function $\mathrm{E}\colon \mathbb{R} \to (0, \infty)$ such that $\frac{g}{f} = \mathrm{E}|_I$ and E is exponential, i.e.

$$\mathrm{E}(x + y) = \mathrm{E}(x)\,\mathrm{E}(y), \qquad x, y \in \mathbb{R};$$

(ii) (multiplicativity) for all $f, g : (a, \infty) \to (0, \infty)$,

$$B_{f \cdot g, k} = B_{f,k} \cdot B_{g,k};$$

(iii) for every $f : (a, \infty) \to (0, \infty)$,

$$B_{\frac{1}{f},k} = \frac{1}{B_{f,k}}.$$

Question

What are properties of the k-variable beta-type mean?

Remark 2

Let $k \in \mathbb{N}$, $k \geq 2$ be fixed. The beta-type mean has the following properties:

(i) $\mathfrak{B}_k$ is homogeneous, i.e.

$$\mathfrak{B}_k\left(tx_1, \ldots, tx_k\right) = t\mathfrak{B}_k\left(x_1, \ldots, x_k\right), \qquad x_1, \ldots, x_k, t > 0.$$

(ii) $\mathfrak{B}_k$ is quasi-arithmetic, i.e. there is a continuous and strictly monotone function $\varphi : (0, \infty) \to \mathbb{R}$ such that

$$\mathfrak{B}_k\left(x_1, \ldots x_k\right) = \varphi^{-1}\left(\frac{\varphi\left(x_1\right) + \ldots + \varphi\left(x_k\right)}{k}\right), \qquad x_1, \ldots, x_k > 0,$$

if, and only if, $k = 2$. Moreover, for $k = 2$, this this equality holds true iff $\varphi(t) = \frac{a}{t} + b$ for some real a, b, $a \neq 0$, and $\mathfrak{B}_2$ is the harmonic mean:

$$\mathfrak{B}_2(x, y) = \frac{2xy}{x + y}, \qquad x, y > 0.$$

A characterization of $\mathfrak{B}_k$ in the class of premeans of beta-type

Using the Krull result on difference equations, employing some convexity condition on f, it is possible to obtain another characterization of beta-type premeans.

Theorem 8.

Let $k \in \mathbb{N}$, $k \geq 2$; and $a \geq 0$ be fixed, and let $I = (a, \infty)$, if $a \geq 0$; or $I = [a, \infty)$, if $a > 0$. Assume that $f : I \to (0, \infty)$ is differentiable and such that the function $\frac{f'}{f} \circ \exp$ is convex. Then the following conditions are equivalent

(i) the beta-type function $B_{f,k}$ is a premean in I;

(ii) there is $c \in \mathbb{R}$ such that

$$f(x) = k^{\frac{1}{(k-1)^2}} \sqrt[k-1]{x} e^{cx}, \qquad x \in I;$$

(iii) $B_{f,k} = \mathfrak{B}_k$.

Krull's theorem on difference equations

Theorem 9.

Let $a \geq -\infty$ be arbitrarily fixed. Suppose that $F : (a, \infty) \to \mathbb{R}$ is convex or concave, and

$$\lim_{x \to \infty} [F(x+1) - F(x)] = 0.$$

Then for every fixed $(x_0, , y_0) \in (a, \infty) \times \mathbb{R}$ there exists exactly one convex or concave function $\varphi : (a, \infty) \to \mathbb{R}$ satisfying the functional equation

$$\varphi(x+1) = \varphi(x) + F(x), \qquad x > a \tag{4}$$

such that

$$\varphi(x_0) = y_0;$$

Theorem 5

moreover, for all $x > a$,

$$\varphi(x) = y_0 + (x - x_0) F(x_0) \tag{5}$$

$$-\sum_{n=0}^{\infty} \{F(x+n) - F(x_0+n) - (x - x_0) [F(x_0+n+1) - F(x_0+n)]\}.$$

A second characterization of $\mathfrak{B}_k$

Applying the theory of iterative functional equations for functions of the class C^n, we obtain another characterization of the k-variable beta-type mean.

Theorem 10.

Let $k \in \mathbb{N}$, $k \geq 2$ be fixed. Assume that $f : (0, \infty) \to (0, \infty)$ is of the class C^2 and the function

$$(0, \infty) \ni x \longmapsto \frac{f(x)}{x^{\frac{1}{k-1}}}$$

has an extension to a class C^2 in the interval $[0, \infty)$.
Then the following conditions are equivalent
(i) the beta-type function $B_{f,k}$ is a premean in $(0, \infty)$;
(ii) there is $c \in \mathbb{R}$ such that

$$f(x) = k^{\frac{1}{(k-1)^2}} \sqrt[k-1]{x} e^{cx}, \qquad x > 0;$$

Affine functions with respect to $\mathfrak{B}_k$

In the this result we determine the functions which are affine with respect to the mean $\mathfrak{B}_k$ for every $k \in \mathbb{N}$, $k \geq 2$.

Theorem 11.

A function $f : (0, \infty) \to (0, \infty)$ is affine with respect to the family of means $\{\mathcal{B}_k : k \in \mathbb{N},\ k \geq 2\}$, i.e.

$$f\left(\mathcal{B}_k\left(x_1, ..., x_k\right)\right) = \mathcal{B}_k\left(f\left(x_1\right), ..., f\left(x_1\right)\right), \qquad x_1, ..., x_k > 0; \ \ k \in \mathbb{N},\ k \geq 2,$$

iff either f is linear, i.e. $f(x) = f(1)\,x$ for all $x > 0$, or f is constant.

The proof relies on the fact that $\mathcal{B}_2 = H$ is the harmonic mean, which is quasi-arithmetic. The affine functions of quasi-arithmetic means are easy to determine. The problem to find all functions $f : (0, \infty) \to (0, \infty)$ which are affine with respect to the beta-type mean $\mathcal{B}_k$ for a fixed $k \in \mathbb{N}$, $k \geq 3$, remains open.

Remark 3

Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}$ be one-to-one and onto. A function $f : I \to \mathbb{R}$ satisfies equation

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) = \varphi^{-1}\left(\frac{\varphi(f(x))+\varphi(f(y))}{2}\right), \qquad x, y \in I,$$

if, and only if, there exist an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$f(x) = \varphi^{-1}(\alpha(\varphi(x)) + b), \qquad x \in I.$$

Remark 4

A function $f : (0, \infty) \to (0, \infty)$ is affine with respect to the mean $\mathfrak{B}_2$, i.e.

$$f\left(\mathfrak{B}_2\left(x, y\right)\right) = \mathfrak{B}_2\left(f\left(x\right), f\left(y\right)\right), \qquad x, y > 0,$$

if, and only if, there exist $p, q \geq 0$, $p + q > 0$, such that

$$f\left(x\right) = \frac{x}{p + qx}, \qquad x > 0.$$

Thank You for your attention

References I

- M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne V. 46, Polish Scientific Publishers, Warszawa, 1968.
- M. Kuczma, B. Choczewski, R. Ger, Iterative Functional Equations, Cambridge University Press, Cambridge, (1990).
- M. Himmel, J. Matkowski, *Homogeneous beta-type functions (J. Class. Anal., Volume 10, Number 1 (2017), 59–66.).*
- M. Himmel, J. Matkowski, *Beta-type means (to appear in J. Difference Equ. Appl.)*
- Young Whan Lee, *Approximate pexiderized gamma-beta type functions*, Journal of Inequalities and Applications 2013, 2013:14.