"Strictly singular operators between $L_p - L_q$ spaces and interpolation "

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Positivity IX Conference, Alberta University

joint work with E. Semenov and P. Tradacete

1. Preliminary

2. Strictly singular non-compact operator sets $V_{p,q}$

3. Extensions of Krasnoselskii compact interpolation theorem to $\mathcal{S}$

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$T : E \to F$ strictly singular $\Leftrightarrow$ for every subspace $E_1 \subset E$ there exists $E_2 \subset E_1$ s.t. $T_{|E_2}$ compact

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• $\mathcal{S}$ is not suitable for interpolation

(some results for operators with the same domain
(*Heinrich* 1980, *Beucher*))

- $K(L^p) = S(L^p) \iff p = 2$

. -Ex: if $p > 2$, $P_{rad}(f) = \sum_n \left(\int_0^1 f(x) r_n(x) dx\right) r_n$

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- $K(L_p, L_q) = S(L_p, L_q)$?

M. Riesz-Thorin convexity theorem

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Theorem (Krasnoselskii 1960)

Let $1 \leq p_0 \,,\, p_1 \,,\, q_0, q_1 \leq \infty$.

If $T : L_{p_0} \to L_{q_0}$ *compact* , $T : L_{p_1} \to L_{q_1}$ *bounded*

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QUESTION :

- extensions of K.T to the class of strictly singular operators ?
- what about endomorphisms ?

-the case of endomorphisms:

Theorem (H.-Semenov-Tradacete, 2010)

Let $1 \leq p \neq q \leq \infty$

If $T : L_p \to L_p$ *bounded ,* $T : L_q \mapsto L_q$ *strictly singular* $\Longrightarrow$
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- **extrapolation** property:

If $1 < q, p < \infty$ and $T : L_q \to L_q$, $T : L_p \to L_p$ bounded.Then
$T \in S(L_r)$ for some $r \in (q, p) \Longleftrightarrow T \in K(L_s)$ for some $s \in (q, p)$.

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Rigidity of special operator classes :
-regular operators in L_p-spaces (Caselles-Gonzalez 1987)
- composition operators on H_p , Volterra operators (Tylli et al. 2016)

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Theorem (Dor, 1975)

Let $1 \le p \ne 2 < \infty$, $0 < \theta \le 1$, *and* $(f_i)_{i=1}^{\infty}$ *in* $L_p[0,1]$. *Assume that either:*

1. $1 \le p < 2$, $\|f_i\| \le 1$ *and* $\|\sum_{i=1}^{n} a_i f_i\| \ge \theta(\sum_{i=1}^{n} |a_i|^p)^{1/p}$ *for scalars* $(a_i)_{i=1}^{n}$, *and every n, or*
2. $2 < p < \infty$, $\|f_i\| \ge 1$ *and* $\|\sum_{i=1}^{n} a_i f_i\| \le \theta^{-1}(\sum_{i=1}^{n} |a_i|^p)^{1/p}$ *for scalars* $(a_i)_{i=1}^{n}$ *and every n.*

Then there exist disjoint measurable sets $(A_i)_{i=1}^{\infty}$ *s.t.*

$$\|f_i \chi_{A_i}\| \ge \theta^{2/|p-2|}.$$

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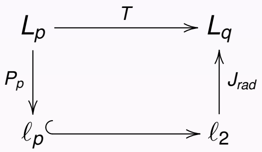
-If $p < 2$, consider T



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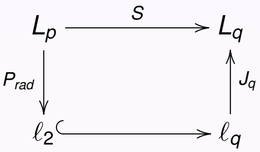

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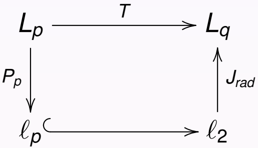

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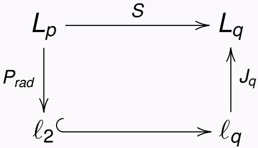

($\star$) If F is a Banach lattice with lower 2-estimate.
If $T : \ell_2 \mapsto F$ strictly singular $\Rightarrow$ compact
(Flores,H.,Kalton,Tradacete 2009)

-If $q \leq 2 \leq p$.

Assume $T \notin K(L_p, L_q)$, there is (x_i) s.t. $\|x_i\|_p = 1$, $x_i \xrightarrow{w} 0$ and $\|Tx_i\|_q \geq \lambda > 0$. By Kadec-Pelczynski m. there is (x_{i_k}) equivalent to the basis of ℓ_p or ℓ_2.

If (x_{i_k}) equivalent to basis of ℓ_p,

$$\lambda n^{\frac{1}{2}} \leq \int_0^1 \Big\| \sum_{k=1}^{n} r_k(t) T x_{i_k} \Big\|_q dt \leq \|T\| \int_0^1 \Big\| \sum_{k=1}^{n} r_k(t) x_{i_k} \Big\|_p dt \leq n^{\frac{1}{p}}.$$

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- If E B. lattice with type 2 and unconditional basis and F B. lattice with a lower 2-estimate $\Rightarrow$ $K(E, F) = S(E, F)$

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• the case $2 < q \le p < \infty$.

If $T : L_p \to L_q$ and $T \in V_{p,q} \Longrightarrow$ there exists normalized (y_k) in L_p s.t.:

- (y_k) is equivalent to the basis of ℓ_2 with $(|y_k|)$ equi-measurable,
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- $[(y_k)]$ is complemented in L_p.

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• There is strictly singular $T : L_p \to L_q$ s. t. T^* is not strictly singular $\iff$ $2 < q < p$.

extrapolation property :

Theorem

Let $1 < p_0, p_1, q_0, q_1 < \infty$ *with* $q_0 \neq q_1$, $p_0 \neq p_1$. *Suppose*

- $\text{mín}\{\frac{q_0}{p_0}, \frac{q_1}{p_1}\} \leq 1$, *or*
- $\text{mín}\{\frac{q_0}{p_0}, \frac{q_1}{p_1}\} > 1$ *and* $\frac{q_1 - q_0}{p_1 - p_0} < 0$.

If $T : L_{p_0} \mapsto L_{q_0}$ *bounded ,* $T : L_{p_1} \mapsto L_{q_1}$ *bounded,*
and $T : L_{p_\theta} \mapsto L_{q_\theta}$ *strictly singular for some* $0 < \theta < 1$

$\Longrightarrow T \in K(L_{p_\tau}, L_{q_\tau})$ *for every* $0 < \tau < 1$.

- Given $T : L_\infty \to L_1$, the V-**characteristic** set ,

$$V(T) := \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) : T \in V_{p,q} = S(L_p, L_q) \backslash K(L_p, L_q) \right\}.$$

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Corollary

$$V(T) \subseteq \partial L(T).$$

interpolation property (extension of Krasnoselskii Thm.):

Theorem

Let $1 \leq p_0, p_1, q_0, q_1 < \infty$.

If $T : L_{p_0} \to L_{q_0}$ *strictly singular* , $T : L_{p_1} \to L_{q_1}$ *bounded* $\Longrightarrow$
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$p_0 < \infty$ is a necessary condition

strong interpolation property :

Proposition

Let $1 < p_0, p_1, q_0, q_1 < \infty$ *with* $p_0 \neq p_1$, $q_0 \neq q_1$

Suppose

- $\text{mín}\{\frac{q_0}{p_0}, \frac{q_1}{p_1}\} \leq 1$,

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Domain restrictions are necessary:

Fixed $0 < \lambda < 1$, consider the segment $S_\lambda = \{(p,q) : \frac{1}{q} = \frac{1}{p} - \lambda\}$

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Take the fractional op. $T_\lambda = T$, for a pairwise disjoint $(A_k)_{k\in\mathbb{N}}$

$$T(f) = \sum_{k\in\mathbb{N}} \left(\frac{1}{\mu(A_k)^{1-\lambda}} \int_{A_k} f d\mu \right) \chi_{A_k}.$$

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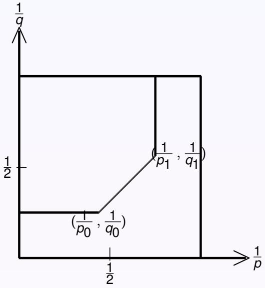
T factors through $\ell_p \hookrightarrow \ell_q$ (so it is strictly singular) , since

$$\begin{array}{ccc} L_p & \xrightarrow{\ T\ } & L_q \\ {\scriptstyle P}\downarrow & & \uparrow{\scriptstyle Q} \\ \ell_p & \xhookrightarrow{\ i\ } & \ell_q \end{array}$$

T is not compact ($T(\mu(A_k)^{-\frac{1}{p}}\chi_{A_k}) = \mu(A_k)^{-\frac{1}{q}}\chi_{A_k}$)

Example: operators $T : L_\infty \mapsto L_1$ with $V(T)$ is



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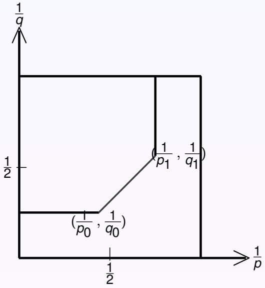

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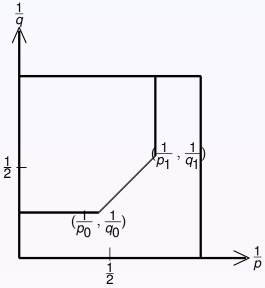

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$$T = T_1 + T_2 + T_3$$

$T_1 : L_p[0, 1/3] \mapsto L_{q_0}[0, 1/3]$ factorizing by $i_{2,q_0} : \ell_2 \to \ell_{q_0}$

$T_2 : L_p[\frac{1}{3}, \frac{2}{3}] \to L_q[\frac{1}{3}, \frac{2}{3}]$ λ-fractional on $[1/3, 2/3]$

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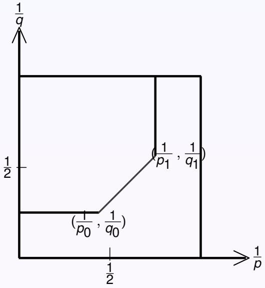

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Question: which is the shape of $V(T)$ in general ??

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$T : X \to Y$ *super strictly singular* (or finitely strictly singular) if it does not exist $c > 0$ and subspaces (F_n) with $dim\, F_n = n$ s.t.

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Bernstein numbers

$$b_n(T) = \sup\left\{ \inf\left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \in F \right\} : F \subset X, \, \mathrm{dim}(F) = n \right\}.$$

T super strictly singular $\iff$ $b_n(T) \underset{n\to\infty}{\longrightarrow} 0$

$T : X \to Y$ super strictly singular $\iff$ ultraoperator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is strictly singular for every ultrafilter $\mathcal{U}$,