# A geometric inequality on the positive cone and an application

Osamu Hatori*

$*$ Supported by JSPS KAKENHI Grant Numbers JP16K05172, JP15K04921, JP15K04897

Niigata University

Positivity IX
17th July, 2017
University of Alberta

# An inequality on the positive cone of $B(H)$

## Theorem 1

*Let $|||\cdot|||$ be a complete uniform norm on $B(H)$. Then for every $t > 0$*

$$|||\log(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}}||| \le |||\log a||| + |||\log b|||$$

*for every $a, b \in B(H)_+^{-1}$.*

When $|||\cdot|||$ is the operator norm $\|\cdot\|$, then the inequality is proved in an elementary way.

A norm $|||\cdot|||$ on $B(H)$ is

- uniform (or symmetric) if $|||ab||| \le |||a|||\cdot\|b\|,\ \|a\|\cdot|||b|||$ for the operator norm $\|\cdot\|$ for every $a, b \in B(H)$

## Theorem 1

*Let $|||\cdot|||$ be a complete uniform norm on $B(H)$.*
*Then for every $t > 0$*

$$|||\log(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}}||| \leq |||\log a||| + |||\log b|||$$

*for every $a, b \in B(H)_+^{-1}$.*

When $|||\cdot|||$ is the operator norm $\|\cdot\|$, then the inequality is proved in an elementary way.

A norm $|||\cdot|||$ on $B(H)$ is

- uniform (or symmetric) if $|||ab||| \leq |||a||| \cdot \|b\|,\ \|a\| \cdot |||b|||$ for the operator norm $\|\cdot\|$ for every $a, b \in B(H)$

It is straight forward that a uniform norm is a unitarily invariant norm.

- untarily invariant if $|||uav||| = |||a|||$ for any unitaries $u$ and $v$ and $a \in B(H)$

$(c, p)$ norm, Ky-Fan norm, the operator norm are all uniform norms. On the matrix algebra a norm is uniform if and only if it is unitarily invariant.
Furthermore A complete uniform norm is equivalent to the operator norm and also satisfies the equalities

$$|||a||| = |||a^*||| = |||\ |a|\ |||$$

for every $a \in B(H)$.

A differential geometric proof applying the inequality of Hiai and Kosaki : $|||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| \le |||\int_0^1 H^s XK^{1-s} ds|||$ [1] gives the inequality described in Theorem 1.

---

[1] Comparison of various means for operators, JFA (1999)

It is straight forward that a uniform norm is a unitarily invariant norm.

- untarily invariant if $|||uav||| = |||a|||$ for any unitaries $u$ and $v$ and $a \in B(H)$

$(c, p)$ norm, Ky-Fan norm, the operator norm are all uniform norms. On the matrix algebra a norm is uniform if and only if it is unitarily invariant.

Furthermore A complete uniform norm is equivalent to the operator norm and also satisfies the equalities

$$|||a||| = |||a^*||| = |||\,|a|\,|||$$

for every $a \in B(H)$.

A differential geometric proof applying the inequality of Hiai and Kosaki : $|||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq |||\int_0^1 H^s XK^{1-s} ds|||$ [1] gives the inequality described in Theorem 1.

[1] Comparison of various means for operators, JFA (1999)

It is straight forward that a uniform norm is a unitarily invariant norm.

- untarily invariant if $|||uav||| = |||a|||$ for any unitaries $u$ and $v$ and $a \in B(H)$

$(c, p)$ norm, Ky-Fan norm, the operator norm are all uniform norms. On the matrix algebra a norm is uniform if and only if it is unitarily invariant.
Furthermore A complete uniform norm is equivalent to the operator norm and also satisfies the equalities

$$|||a||| = |||a^*||| = |||\ |a|\ |||$$

for every $a \in B(H)$.

A differential geometric proof applying the inequality of Hiai and Kosaki : $|||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq |||\int_0^1 H^s XK^{1-s} ds|||$ [1] gives the inequality described in Theorem 1.

[1] Comparison of various means for operators, JFA (1999)

It is straight forward that a uniform norm is a unitarily invariant norm.

- untarily invariant if $|||uav||| = |||a|||$ for any unitaries $u$ and $v$ and $a \in B(H)$

$(c, p)$ norm, Ky-Fan norm, the operator norm are all uniform norms. On the matrix algebra a norm is uniform if and only if it is unitarily invariant.
Furthermore A complete uniform norm is equivalent to the operator norm and also satisfies the equalities

$$|||a||| = |||a^*||| = |||\ |a|\ |||$$

for every $a \in B(H)$.

A differential geometric proof applying the inequality of Hiai and Kosaki : $|||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq ||| \int_0^1 H^s XK^{1-s} ds|||$ [1] gives the inequality described in Theorem 1.

[1] Comparison of various means for operators, JFA (1999)

# Isometries on positive cones

## Problem

*Suppose that $U : G_1 \to G_2$ is a surjection between positive cones ($G_j \subset B(H_j)_+^{-1}$) which preserve a certain distance mesures.*

- *the form of $U$?*
- *the propety of $U$?*
- *the group of all of $U$?*

- $\| \log a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \|$ : Molnár (2009, PAMS), Molnár and Nagy (2010 EJLA), H. and Molnár (2014, JMAA)
- $|||\log M^{-\frac{1}{2}} N M^{-\frac{1}{2}}|||$ : Molnár (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LMA)
- $|||f(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})|||$ : Molnár and Szokol (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LAA) and Molnár (2015, Oper. Th. Adv. Appl. 250)
- Bregman and Jensen divergences : Molnár, Pitrik and Virosztek (2016, LAA)
- quasi-entropies : Molnár (2017, JMAA)

## Problem

*Suppose that $U : G_1 \to G_2$ is a surjection between positive cones ($G_j \subset B(H_j)_+^{-1}$) which preserve a certain distance mesures.*

- *the form of $U$?*
- *the propety of $U$?*
- *the group of all of $U$?*

- $\| \log a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \|$ : Molnár (2009, PAMS), Molnár and Nagy (2010 EJLA), H. and Molnár (2014, JMAA)
- $||| \log M^{-\frac{1}{2}} N M^{-\frac{1}{2}} |||$ : Molnár (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LMA)
- $|||f(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})|||$ : Molnár and Szokol (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LAA) and Molnár (2015, Oper. Th. Adv. Appl. 250)
- Bregman and Jensen divergences : Molnár, Pitrik and Virosztek (2016, LAA)
- quasi-entropies : Molnár (2017, JMAA)

## Problem

*Suppose that $U : G_1 \to G_2$ is a surjection between positive cones ($G_j \subset B(H_j)_+^{-1}$) which preserve a certain distance mesures.*

- *the form of $U$?*
- *the propety of $U$?*
- *the group of all of $U$?*

- $\| \log a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \|$ : Molnár (2009, PAMS), Molnár and Nagy (2010 EJLA), H. and Molnár (2014, JMAA)
- $||| \log M^{-\frac{1}{2}} N M^{-\frac{1}{2}} |||$ : Molnár (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LMA)
- $|||f(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})|||$ : Molnár and Szokol (for $\mathbb{M}_n(\mathbb{C})_+^{-1}$ 2015, LAA) and Molnár (2015, Oper. Th. Adv. Appl. 250)
- Bregman and Jensen divergences : Molnár, Pitrik and Virosztek (2016, LAA)
- quasi-entropies : Molnár (2017, JMAA)

## Theorem 2

*$G_j = \exp E_j$ for $E_j$, a real linear subspace of $B(H_j)_S$ such that $aba \in G_j$ for every pair $a, b \in G_j$.*
*Suppose : $U : G_1 \to G_2$ is a surjection such that*

$$|||\log(a^{-\frac{t}{2}} b^t a^{-\frac{t}{2}})^{\frac{1}{t}}|||_1 = \|\log(U(a)^{-\frac{t}{2}} U(b)^t U(a)^{-\frac{t}{2}})^{\frac{1}{t}}\||_2, \quad a, b \in G_1.$$

$\Longrightarrow$
*$\exists f : E_1 \to E_2$ (bijection, commutativity preserving linear map in both directions, isometry ) with such that*

$$U_0(a) = \exp(f(\log a)), \quad a \in G_1,$$

$$U(a) = U(e) \oplus_t U_0(a), \quad a \in G_1,$$

$$U_0(aba) = U_0(a)U_0(b)U_0(a), \quad a, b \in G_1.$$

Note that $U_0$ is a continuous Jordan isomorphism.

Examples of $E \subset A$ such that $aba \in \exp E$ for any $a, b \in \exp E$

## Example

$E = \mathbb{H}_n(\mathbb{C}) \subset \mathbb{M}_n(\mathbb{C})$
Then $\exp \mathbb{H}_n(\mathbb{C}) = \mathbb{P}_n$ : the set of all positive definite complex matrices

## Example

$E = B(H)_S$ : the space of self-adjoint elements in $B(H)$
Then $\exp B(H)_S = B(H)_+^{-1}$ : the set of all positive invertible elements in $B(H)$
Then $B(H)_+^{-1} =$

## Example

$E = A_S$ for a unital $C^*$-algebra $A$
Then $\exp A_S = A_+^{-1}$ : the set of all positive invertible elemets in $A$.

We prove Theorem 2 by applying a Mazur-Ulam theorem for a generalized gyrovector space = GGV.

The celebrated Mazur-Ulam theorem is

**The Mazur-Ulam theorem**

*A surjective isometry between normed linear spaces is affine=linear + constant.*

Our Mazur-Ulam therem is for GGV. Applying the inequality in Theorem 1 we prove that certain positive cones are GGV. Then we can prove Theorem 2.

# A gyrogroup and a GGV

## Definition 1 ((Gyrocommutative) Gyrogroup)

*A groupoid* $(G, \oplus)$ *is a gyrogroup if there exists a point* $\boldsymbol{e} \in G$ *such that the following hold.*

(G1) $\forall \boldsymbol{a} \in G$

$$\boldsymbol{e} \oplus \boldsymbol{a} = \boldsymbol{a}, \quad .$$

(G2) $\forall \boldsymbol{a} \in G \;\; \exists \ominus \boldsymbol{a} \;\; s.t.$

$$\ominus \boldsymbol{a} \oplus \boldsymbol{a} = \boldsymbol{e}.$$

(G3) $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G \;\; \exists \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G \quad s.t.$

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}.$$

(G4) $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]$ *is an gyroautomorphism for* $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

(G5) $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

$$\mathrm{gyr}[\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{b}] = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}].$$

*Gyrocommutative if the following (G6) is also satisfied.*

(G6) $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

$$\boldsymbol{a} \oplus \boldsymbol{b} = \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a}).$$

$\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}] = \mathrm{Id} \quad \forall a, b \Rightarrow (G, \oplus)$ is a (commutative) group.

A (gyrocommutative) gyrogroup is a generalization of an (Abelian) group.

## Definition 1 ((Gyrocommutative) Gyrogroup)

*A groupoid* $(G, \oplus)$ *is a gyrogroup if there exists a point* $\boldsymbol{e} \in G$ *such that the following hold.*

(G1) *Existing of unit*

(G2) *Existing of the inverse element for each element.*

(G3) *Not necessarily associative, but "weakly associative".*

(G4)

(G5)

*Gyrocommutative if the following (G6) is also satisfied.*

(G6) *Not necessarily commutative, but "weakly commutative".*

$\mathbb{R}^3_c = \{\boldsymbol{u} \in \mathbb{R}^3 : \|\boldsymbol{u}\| < c\}$ : the set of all admissible velocities in Einstein's theory of special relativity, where $c$ is the speed of light in vacuum.

The Einstein velocity addition $\oplus_E$ in $\mathbb{R}^3_c$ is given by

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v} \rangle / c} \left\{ \boldsymbol{u} + \frac{1}{\gamma_{\boldsymbol{u}}} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_{\boldsymbol{u}}}{1 + \gamma_{\boldsymbol{u}}} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3_c$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $\gamma_{\boldsymbol{u}}$ is the Lorentz factor given by

$$\gamma_{\boldsymbol{u}} = (1 - \|\boldsymbol{u}\|^2 / c^2)^{-\frac{1}{2}}.$$

Then $(\mathbb{R}^3_c, \oplus_E)$ is not a group but a gyrocommutative gyrogroup.

## Lemma

*$E \subset B(H)$ : real linear subspace s.t.*
*$aba \in \exp E$ for any $a, b \in \exp E$ : $\exp E$ is closed under the formation of the Jordan product*

$\Longrightarrow$

*$\exp E$ is a gyrocommutative gyrogroup with*
*$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}, \quad a, b \in G$ for $t > 0$.*

The gyrogroup identity $=$ the identity element $e = \exp 0$ .
The inverse element $\ominus a$ is $a^{-1}$
For $a, b \in A_+^{-1}$ put

$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$

Then $X$ is a unitary and

$$\mathrm{gyr}[a, b]c = XcX^*, \quad a, b, c \in \exp E.$$

## Definition 3 (generalized gyrovector space; GGV )

A gyrocommutative gyrogroup $(G, \oplus)$ is a GGV if $\otimes : \mathbb{R} \times G \to G$, and an injection $\phi : G \to (\mathbb{V}, \|\cdot\|)$ are defined, where $(\mathbb{V}, \|\cdot\|)$ is a real normed space.

(GGV0) $\|\phi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a})\| = \|\phi(\boldsymbol{a})\| \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$;

(GGV1) $1 \otimes \boldsymbol{a} = \boldsymbol{a} \quad \forall \boldsymbol{a} \in G$;

(GGV2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a}) \quad \forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a}) \quad \forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV4) $(\phi(|r| \otimes \boldsymbol{a}))/\|\phi(r \otimes \boldsymbol{a})\| = \phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|$
$\forall \boldsymbol{a} \in G \setminus \{\boldsymbol{e}\},\ r \in \mathbb{R} \setminus \{0\}$;

(GGV5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R}$;

(GGV6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G \quad \forall \boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGVV) $\{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\|\phi(r \otimes \boldsymbol{a})\| = |r| \otimes' \|\phi(\boldsymbol{a})\| \quad \forall \boldsymbol{a} \in G, r \in R$;

(GGV8) $\|\phi(\boldsymbol{a} \oplus \boldsymbol{b})\| \leq \|\phi(\boldsymbol{a})\| \oplus' \|\phi(\boldsymbol{b})\| \quad \forall \boldsymbol{a}, \boldsymbol{b} \in G$.

## Definition 3 ( gyrovector space defined by Ungar)

A gyrocommutative gyrogroup $(G, \oplus)$ is a gyrovector space if $\otimes : \mathbb{R} \times G \to G$ is defined, where $(\mathbb{V}, \|\cdot\|)$ is a real inner product space and $G \subset \mathbb{V}$.

(GGV0) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \cdot \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G;$

(GGV1) $1 \otimes \boldsymbol{a} = \boldsymbol{a} \quad \forall \boldsymbol{a} \in G;$

(GGV2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a}) \quad \forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R};$

(GGV3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a}) \quad \forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R};$

(GGV4) $(|r| \otimes \boldsymbol{a})/\|r \otimes \boldsymbol{a}\| = \boldsymbol{a}/\|\boldsymbol{a}\| \quad \forall \boldsymbol{a} \in G \setminus \{\boldsymbol{e}\},\ r \in \mathbb{R} \setminus \{0\};$

(GGV5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R};$

(GGV6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G \quad \forall \boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R};$

(GGVV) $\{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\|r \otimes \boldsymbol{a}\| = |r| \otimes' \|\boldsymbol{a}\| \quad \forall \boldsymbol{a} \in G, r \in R;$

(GGV8) $\|\boldsymbol{a} \oplus \boldsymbol{b}\| \leq \|\boldsymbol{a}\| \oplus' \|\boldsymbol{b}\| \quad \forall \boldsymbol{a}, \boldsymbol{b} \in G.$

# A brief look at GGV

GGV is an exotic normed space
defined on a gyrocommutative gyrogroup.

| normed space | GGV |
|---|---|
| commutative group | gyrocommutative gyrogroup |
| scalar multiplication | scalar multiplication’ |
| norm | norm’ |

A Normed space is a GGV, but a GGV is sometimes far from beeing a linear space in general.

The positive cone of a unital $C^*$-algebra is GGV

# A brief look at GGV

GGV is an exotic normed space
defined on a gyrocommutative gyrogroup.

| normed space | GGV |
|---|---|
| commutative group | gyrocommutative gyrogroup |
| scalar multiplication | scalar multiplication’ |
| norm | norm’ |

A Normed space is a GGV, but a GGV is sometimes far from beeing a linear space in general.

The positive cone of a unital $C^*$-algebra is GGV

# A brief look at GGV

GGV is an exotic normed space
defined on a gyrocommutative gyrogroup.

| normed space | GGV |
|---|---|
| commutative group | gyrocommutative gyrogroup |
| scalar multiplication | scalar multiplication' |
| norm | norm' |

A Normed space is a GGV, but a GGV is sometimes far from beeing a linear space in general.

The positive cone of a unital $C^*$-algebra is GGV

# A brief look at GGV

GGV is an exotic normed space
defined on a gyrocommutative gyrogroup.

| normed space | GGV |
|---|---|
| commutative group | gyrocommutative gyrogroup |
| scalar multiplication | scalar multiplication’ |
| norm | norm’ |

A Normed space is a GGV, but a GGV is sometimes far from beeing a linear space in general.

The positive cone of a unital $C^*$-algebra is GGV

## Example

$E \subset B(H)_S$ such that $aba \in \exp E$ for $\forall a, b \in \exp E$
$\exp E$ : the gyrocommutative gyrogroup with

$$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}, \qquad a, b \in A_+^{-1}.$$

Let $|||\cdot|||$ be a complete uniform norm on $B(H)$. Put

$$r \otimes a = a^r, \quad \phi(a) = \log a, \qquad a \in \exp E,\ r \in \mathbb{R},$$

$$(\pm|||\log(\exp E)|||, \oplus', \otimes') = (\mathbb{R}, +, \times).$$

$\Longrightarrow$

$(\exp E, \oplus_t, \otimes, \log)$ is a GGV with $|||\cdot|||$
In particular, $(A_+^{-1}, \oplus_t, \otimes, \log)$ is a GGV with $|||\cdot|||$.

### Theorem (A Mazur-Ulam theorem for GGV (Abe and H.))

*Suppose that* $U : (G_1, \oplus_1, \otimes_1, \varphi_1) \to (G_2, \oplus_2, \otimes_2, \varphi_2)$ *is a surjection. Then*

$$\|\varphi_2(\ominus_2 U(\boldsymbol{a}) \oplus_2 U(\boldsymbol{b}))\|_2 = \|\varphi_1(\ominus_1 \boldsymbol{a} \oplus_1 \boldsymbol{b})\|_1, \quad \forall \boldsymbol{a}, \boldsymbol{b} \in G$$

$\Longleftrightarrow$

$$U(\boldsymbol{a}) = U(\boldsymbol{e}) \oplus_2 U_0(\boldsymbol{a}), \qquad \forall \boldsymbol{a} \in G,$$

*where* $U_0$ *is an isometrical isomorphism :*
$U_0 : G_1 \to G_2$ *is a bijection s.t.* $\forall \boldsymbol{a}, \boldsymbol{b} \in G_1$, $\forall \alpha \in \mathbb{R}$

(1) $$U_0(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = U_0(\boldsymbol{a}) \oplus_2 U_0(\boldsymbol{b});$$

(2) $$U_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 U_0(\boldsymbol{a});$$

(3) $$\varrho_2(U_0\boldsymbol{a}, U_0\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}).$$

In the case where $(G_j, \oplus_j, \otimes_j, \varphi_j)$ is a usual normed space, then the above theorem is just the celebrated Mazur-Ulam theorem.

Let $G_j = \exp E_j$ for $E_j$, a real linear subspace of $B(H_j)_S$ such that $aba \in G_j$ for every pair $a, b \in G_j$.

## Theorem 2

*Suppose that $U : G_1 \to G_2$ is a surjection such that*

$$\||\log(a^{-\frac{t}{2}}b^t a^{-\frac{t}{2}})^{\frac{1}{t}}\||_1 = \|\log(U(a)^{-\frac{t}{2}}U(b)^t U(a)^{-\frac{t}{2}})^{\frac{1}{t}}\||_2, \quad a, b \in G_1.$$

$\Longrightarrow$
*$\exists f : E_1 \to E_2$ (bijection, commutativity preserving linear map in both directions, isometry ) such that*

$$U_0(a) = \exp(f(\log a)), \quad a \in G_1,$$

$$U(a) = U(e) \oplus_t U_0(a), \quad a \in G_1,$$

$$U_0(aba) = U_0(a)U_0(b)U_0(a), \quad a, b \in G_1.$$

Thank you for your time!