The Itô integral for martingales in vector lattices

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Motivation

In the classical setting of measure spaces, the Itô integral is closely related to the Doob-Meyer decomposition of submartingales:

- If (X_t) is a submartingale, then (under certain assumptions),

$$X_t = M_t + A_t,$$

  where (M_t) is a martingale, (A_t) is an increasing system and the decomposition is unique.
- The measure involved with accompanying integral that yields the “Itô integral" is determined by the increasing system (A_t).

Motivation

What is this measure and accompanying integral?

Notation and definitions

- We use the same notation, definitions and assumptions as in the preceding lecture by Koos Grobler.
- A *stochastic process* in $\mathfrak{E}$ is a function $t \mapsto X_t \in \mathfrak{E}$, for $t \in J$, with $J \subset \mathbb{R}^+$ an interval.
- The stochastic process $(X_t)_{t\in J}$ is *adapted to the filtration* if $X_t \in \mathfrak{F}_t$ for all $t \in J$.

Definition

A stochastic process $(A_t)_{t\in J}$ is called an *adapted increasing process* if

1. $(A_t)_{t\in J}$ is adapted to the filtration $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in J}$;
2. $A_a = 0$ if a is the left endpoint of J and $A_s \leq A_t$ for $s \leq t$, $s, t \in J$.
3. $(A_t)_{t\in J}$ is right-continuous, i.e., $A_t \downarrow A_s$ as $t \downarrow s$.

If $J = [a, \infty)$ the adapted increasing process (A_t) is called *integrable* if $A_\infty := \sup_{t\in J} A_t \in \mathfrak{E}$.

Given an increasing right-continuous bounded process $0 \leq A_t$ in $\mathfrak{E}$, we define a Stieltjes-Lebesgue measure μ_A on the algebra $\mathcal{F}(J)$ generated in J by the left-open and right-closed intervals as follows:

1. $\mu_A(a, b] := A_b - A_a,\ a, b \in J.$
2. For any disjoint union $\bigcup_{k=1}^{n} I_k$ of left-open and right-closed intervals,
$$\mu_A\Big(\bigcup_{k=1}^{n} I_k\Big) := \sum_{k=1}^{n} \mu_A(I_k).$$

We can define $\mu_A(\emptyset) = 0$, but it actually follows from $\emptyset = (a, a]$.

Theorem

Let $(A_t)_{t\in J}$ *be an integrable adapted increasing process. Then* μ_A *is an order countably additive* $\mathfrak{E}$*-valued measure on the algebra* $\mathcal{F}(J)$ *of all finite unions of left-open and right-closed intervals in* J*. This means that if* (E_n) *is a sequence of disjoint subsets in* $\mathcal{F}(J)$ *such that* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}(J)$ *then*

$$\sum_{n=1}^{k} \mu_A(E_n) \uparrow \mu_A(\bigcup_{n=1}^{\infty} E_n).$$

- The next step is to extend this measure to the Borel σ-algebra $\mathcal{B}(J)$ generated by $\mathcal{F}(J)$.
- The steps are exactly that of the Carathéodory extension procedure for extending a real-valued measure.

Theorem

Let $\mathfrak{E}$ be a Dedekind complete Riesz space separated by its order continuous dual $\mathfrak{E}^{\sim}_{00}$. If $(A_t)_{t\in J}$ is an integrable adapted increasing process, then μ_A can be extended uniquely to a countably additive $\mathfrak{E}$-valued measure on the $sigma$-algebra $\mathcal{B}(J)$ of all Borel subsets of J.

Vector integration of vector-valued functions

- Throughout the remainder of the presentation our assumptions will be that $\mathfrak{E}$ is a Dedekind complete vector lattice with a weak order unit E, with separating order continuous dual $\mathfrak{E}_{00}^{\sim}$ and with a conditional expectation $\mathbb{F}$ defined on $\mathfrak{E}$, satisfying $\mathbb{F}(E) = E$.
- In addition, we will assume that $\mathfrak{E} = (\mathfrak{E}_{00}^{\sim})_{00}^{\sim}$, i.e., we will assume that $\mathfrak{E}$ is a perfect Riesz space, and that $\mathfrak{E}$ is $\mathbb{F}$-universally complete in $\mathfrak{E}^u$.

- Let Φ be the set of all elements $\phi \in \mathfrak{E}_{00}^{\sim}$ such that $|\phi|(E) = 1$.
- We note that $\mathbb{F}$ and $|\phi|$ can be extended to $\mathfrak{E}_s$ (the proof for the functional $|\phi|$ follows in the same manner as the proof for $\mathbb{F}$) and therefore, allowing the value $+\infty$, the seminorms we are about to define make sense for all elements of $\mathfrak{E}_s$.

The following constructions and facts are to be found in [20].

- For $\phi \in \Phi$, we define the Riesz seminorm

$$p_\phi(X) := |\phi|(\mathbb{F}(|X|))$$

  and denote the set of all these seminorms by $\overline{P}$.
- Similarly, for such ϕ we define the Riesz seminorm

$$q_\phi(X) := (|\phi|(\mathbb{F}(|X|^2)))^{1/2},$$

  where for $X \in \mathfrak{E}$ the product is formed in $\mathfrak{E}^u$.
- We denote the set of all these seminorms by $\overline{Q}$.
- All these seminorms are order continuous Riesz seminorms.

- The space $\mathcal{L}^1$ is defined to be the space $(\mathfrak{E}, \sigma(\overline{P}))$ and we have that $\mathcal{L}^1$ is the set of all $X \in \mathfrak{E}_s$ such that $p_\phi(X) < \infty$ for all $p_\phi \in \overline{P}$, equipped with the weak topology $\sigma(\overline{P})$.
- The proof of this fact in [20, Section 3] depends on the assumption that $\mathfrak{E} = \mathrm{dom}\mathbb{F}$, which holds in this case by our assumption that $\mathfrak{E}$ is $\mathbb{F}$-universally complete in $\mathfrak{E}^u$.
- Similar to the case of $\mathfrak{L}^1$, it follows that if $\mathcal{L}^2$ is the set of all $X \in \mathfrak{E}_s$ such that $q_\phi(X) < \infty$ for all $q_\phi \in \overline{Q}$ equipped with the topology $\sigma(\overline{Q})$, then $\mathcal{L}^2 = \{X \in \mathfrak{E}_s \,:\, |X|^2 \in \mathfrak{E} = \mathcal{L}^1\}$.

As noted in [19], a standard computation with the bilinear form $\langle X, Y\rangle_\phi := \phi\mathbb{F}(XY)$ defined on $\mathcal{L}^2 \times \mathcal{L}^2$ yields the Cauchy inequality

$$p_\phi(XY) \le q_\phi(X)q_\phi(Y), \text{ for all } X, Y \in \mathcal{L}^2. \qquad \text{(CS)}$$

- The spaces $\mathcal{L}^1$ and $\mathcal{L}^2$ with their respective topologies are topologically complete and the topologies are Lebesgue topologies (see [1] and [20]).
- Both p_ϕ and q_ϕ, restricted to the carrier band of ϕ, are norms.
- We need to show that they are complete norms as required in the definition of the Dobrakov integral given below.
- We need the following result.

Lemma

Let ψ be a strictly positive order continuous linear functional on $\mathfrak{E}$. If $0 \leq X \in \mathfrak{E}_s$ and if the extension of ψ to $\mathfrak{E}_s$ satisfies $\psi(X) < \infty$, then $X \in \mathfrak{E}^u$.

Theorem

Under the assumptions stated in the beginning of the section, the norms p_ϕ and q_ϕ restricted to the carrier bands of ϕ are complete norms.

The spaces

$$\mathcal{L}^1_\phi := \{\mathbb{P}_\phi X \in \mathfrak{E}_s : p_\phi(X) < \infty\}$$

and

$$\mathcal{L}^2_\phi := \{\mathbb{P}_\phi X \in \mathfrak{E}_s : q_\phi(X) < \infty\}$$

are therefore Banach lattices (even a Hilbert lattice in the case of q_ϕ).

Dobrakov integral

- We need an integral for vector valued functions relative to a vector measure.
- Two such integrals are known in the literature, namely the *Bartle integral* ([5, 9]) and the *Dobrakov integral* ([10]).
- For countably additive measures the latter integral is the more general one and we shall use it in the sequel.
- The Dobrakov integral is defined for functions having values in a Banach space G, and a measure that maps sets into the space $L(G,H)$ of all bounded linear operators from G into a Banach space H, where the measure is countably additive in the strong operator topology on $L(G,H)$.

Special case

- We shall firstly consider the case in which we have a fixed strictly positive order continuous linear functional $0 \leq \phi \in \mathfrak{E}^{\sim}_{00}$ on $\mathfrak{E}$. In this case the spaces $\mathcal{L}^1_\phi$ and $\mathcal{L}^2_\phi$ are Banach spaces (and the latter is of course a Hilbert space).
- If $(A_t)_{t \in J}$ is an integrable increasing right-continuous process, we have the vector measure μ_A on the Borel subsets $\mathcal{B} = \mathcal{B}(J)$ of $J = [a, b]$ and we will assume its values to be in $\mathcal{L}^2_\phi$.
- The stochastic processes we want to integrate will also be assumed to take values in $\mathcal{L}^2_\phi$ and so their product (in the f-algebra $\mathfrak{E}^u$) will have values in $\mathcal{L}^1_\phi$.

- Define, for any $S \in \mathcal{B}$, the multiplication operator $T_S : \mathcal{L}^2_\phi \to \mathcal{L}^1_\phi$, corresponding to μ_A, by $T_S X = \mu_A(S)X$.
- The map $S \mapsto T_S$ is then an $L(\mathcal{L}^2_\phi, \mathcal{L}^1_\phi)$-valued measure, where $L(\mathcal{L}^2_\phi, \mathcal{L}^1_\phi)$ denotes the space of all continuous linear operators from $\mathcal{L}^2_\phi$ into $\mathcal{L}^1_\phi$ and $\|T_S\| = q_\phi(\mu_A(S))$.
- Since μ_A is countably additive in order, we have for each disjoint sequence of sets (S_i) that

$$|\sum_{i=1}^{n} \mu_A(S_i) - \mu_A(\bigcup_{i=1}^{\infty} S_i)| \to 0$$

in order, and since q_ϕ is order continuous, the convergence is also in the norm q_ϕ.

- This means that

$$\| \sum_{i=1}^{n} T_{S_i} - T_{\bigcup_{i=1}^{\infty} S_i} \| \to 0.$$

- Thus, the operator valued measure is uniformly (and therefore strongly) σ-additive.
- We identify the measure μ_A with the operator valued measure $S \mapsto T_S$.

- We shall adapt Dobrakov's definition of integrability slightly to take into account the fact that we have a lattice structure.
- This will have the advantage that we can prove a Lebesgue dominated convergence theorem, something the Dobrakov integral in general lacks.
- For the benefit of the reader we will recall the relevant definitions of measurability and integrability.
- Finally we will proceed to define the integral for the case where we do not assume the existence of a strictly positive linear functional.

The following facts and definitions are necessary to define the integral:

- The *ϕ-semivariation of μ_A* is defined as

$$\hat{\mu}_A(S) = \sup p_\phi \left(\sum_{i=1}^{r} X_i \mu_A(S \cap S_i) \right)$$

  the supremum taken over all measurable partitions (S_i) of $[a,b]$ and all $X_i \in \mathcal{L}^2_\phi$ with $q_\phi(X_i) \leq 1$. We denote by $\mathcal{B}_0$ the class of all sets in $\mathcal{B}$ with finite ϕ-semivariation and by $\sigma(\mathcal{B}_0)$ the σ-algebra generated in $\mathcal{B}$ by $\mathcal{B}_0$.
- A *measurable step function* is a function of the form

$$X_t = \sum_{i=1}^{k} X_i I_{S_i}(t),\ \ S_i \in \mathcal{B}, S_i \cap S_j = \emptyset \text{ for } i \neq j, X_i \in \mathcal{L}^2_\phi.$$

- The function $t \mapsto X(t)$ is called *$\mathcal{B}$-measurable* if there exists a sequence $(X_n(t))$ of measurable step functions that converges in $\mathcal{L}^2_\phi$ to $X(t)$ for every $t \in J$.

- A measurable step function $X_t = \sum_{i=1}^{r} X_i I_{S_i}(t)$ is called a μ_A-*integrable step function* if $\hat{\mu}_A(S_i) < \infty$ for all i. Thus a measurable step function is μ_A-integrable if it is a $\mathcal{B}_0$-simple function (i.e., each $S_i \in \mathcal{B}_0$).
- A function $X(t)$ is called μ_A-*measurable* if there exists a sequence $(X_n(t))$ of μ_A-integrable step functions that converges to $X(t)$ in every point t.
- We call a set $N \in \mathcal{B}$ a μ_A-null set if $\hat{\mu}_A(N) = 0$ and μ_A-almost convergence of a sequence means convergence in all points except in those belonging to a μ_A-null set.
- The *integral* of a measurable step function $X_t = \sum_{i=1}^{r} X_i I_{S_i}(t)$ is defined as

$$\int_S X \, d\mu_A := \sum_{i=1}^{r} X_i \mu_A(S \cap S_i), \; S \in \mathcal{B}.$$

- A sequence of vector measures (μ_k) with values in a Banach space is called *uniformly countably additive* whenever, for every $\epsilon > 0$ and every sequence of sets $S_n \downarrow \emptyset$ in $\mathcal{B}$ there exists some $n_0 \in \mathbb{N}$ such that $\sup_k \|\mu_k(S_n)\| < \epsilon$ for all $n \geq n_0$.

We note that if $X_t = \sum_{i=1}^r X_i I_{S_i}(t)$ is a measurable step function, then, since $|X_t| = \sum_{i=1}^r |X_i| I_{S_i}(t)$, we have that $|X_t|$ is also a measurable step function. Moreover,

$$\left| \int_S X \, d\mu_A \right| \leq \sum_{i=1}^r |X_i| \mu_A(S \cap S_i) = \int_S |X| \, d\mu_A, \; S \in \mathcal{B}.$$

Definition

The μ_A-measurable function X is said to be (Dobrakov) integrable if

1. there exists a sequence (X_n) of μ_A-integrable step functions that converges μ_A-almost everywhere to X;
2. the sequence of set functions $(|\nu|_n)_{n=1}^{\infty}$ defined by

$$|\nu|_n(S) = \int_S |X_n| \, d\mu_A, \; S \in \mathcal{B},$$

   is uniformly σ-additive on $\mathcal{B}$. We call (X_n) a defining sequence for X.

- The fact that $|\int_S X_n \, d\mu_A| \leq \int_S |X_n| \, d\mu_A$ for all $S \in \mathcal{B}$, implies that for the sequence (X_n) in the definitions above, we have that the sequence of set functions

$$\nu_n(S) = \int_S X_n \, d\mu_A, \; S \in \mathcal{B},$$

  is also uniformly σ-additive.
- In the Dobrakov integration theory a function X is called integrable if the first condition above holds and if instead of the second condition one has that the sequence $(\nu_n)_{n=1}^{\infty}$ is uniformly σ-additive.
- Thus our condition of integrability implies that of Dobrakov.

Lemma

Using the notation above we have:

1. *If X is integrable then $|X|$ is integrable.*
2. *For each $S \in \mathcal{B}$ the limits*

$$\nu(S) := \lim_{n\to\infty} \int_S X_n \, d\mu_A \text{ and } |\nu|(S) := \lim_{n\to\infty} \int_S |X_n| \, d\mu_A$$

   exist and $|\nu(S)| \leq |\nu|(S)$.

3. *The limits are independent of the choice of (X_n) and are uniform in S.*

Definition

With the definitions and notation as above, we define the Dobrakov integral of an integrable function $X : J \to \mathcal{L}^2_\phi$ with defining sequence (X_n) as

$$\int_S X \, d\mu_A = \lim_{n\to\infty} \int_S X_n \, d\mu_A, \; S \in \mathcal{B}.$$

Proposition

(1) *We have*

$$\int_S |X|\,d\mu_A = \lim_{n\to\infty}\int_S |X_n|\,d\mu_A,\ S\in\mathcal{B}$$

for every defining sequence (X_n) *of* X.

(2) *If* $0\le X$ *we have* $\int_S X\,d\mu_A \ge 0$ *for all* $S\in\mathcal{B}$. *Thus, if* X *and* Y *are integrable, and if* $X\le Y$, *then* $\int_S X\,d\mu_A \le \int_S Y\,d\mu_A$ *for all* $S\in\mathcal{B}$.

(3) *If* $X\ge 0$ *then there exists a defining sequence of* X *consisting of positive integrable functions.*

(4) *If* X *is integrable, then*

$$\left|\int_S X\,d\mu_A\right| \le \int_S |X|\,d\mu_A \text{ for all } S\in\mathcal{B}.$$

- We have that integrability of X implies that of $|X|$.
- The converse is also true and it follows from the following results that show that the integrable functions is an ideal in the space of measurable functions.

Proposition

If X is a measurable function and if $|X| \leq Y$ with Y an integrable function, then X is integrable. In particular, if X is measurable and if $|X|$ is integrable, then X is also integrable.

- In the general case of Banach spaces a direct analogue of Lebesgue's dominated convergence theorem is not readily available.
- However, in the case of lattices with our definition of integrability, we have a Lebesgue theorem that is easy to derive from Dobrakov's theory.

Theorem

Let (X_n) be a sequence of integrable functions converging μ_A-almost everywhere to a measurable function X. Suppose that there exists an integrable function Y such that $|X_n| \leq Y$. Then X is integrable and

$$\int_S X\, d\mu_A = \lim_{n\to\infty} \int_S X_n\, d\mu_A, \; S \in \mathcal{B}.$$

This limit is uniform with respect to $S \in \mathcal{B}$.

General case

- Having defined the integral for the case that we have a strictly positive order continuous linear functional, we now define the integral for the general case: Let therefore $\mathcal{L}^1$ and $\mathcal{L}^2$ be the locally solid spaces with topologies generated by the sets of Riesz seminorms P and Q respectively.
- We define $\mathcal{B}_0$ to be the class of all sets $S \in \mathcal{B}$ such that $(\hat{\mu}_A)_\phi(S) < \infty$ for every order continuous $\phi \geq 0$ satisfying $\phi(S) = 1$.
- An *integrable simple function* is then a ϕ-integrable function for all ϕ, i.e., it is a $\mathcal{B}_0$-simple function.

- A function $X : J \to \mathcal{L}^2$ is called *measurable* if there exists a sequence (X_n) of integrable simple functions such that $q_\phi(X_n(t) - X(t)) \to 0$ for every $q_\phi \in Q$.
- A measurable function $X : J \to \mathcal{L}^2$ is called *integrable* if there exists a sequence of simple integrable functions (X_n) converging μ_A-almost everywhere to X in the $\sigma(\mathcal{L}^2, Q)$ topology and for which the sequence of set functions $(|\nu|_n(S))$ defined by $|\nu|_n(S) = \int_S |X_n|\, d\mu_A$ is uniformly σ-additive in $\mathcal{B}$ with reference to the topology $\sigma(\mathcal{L}^1, P)$.
- As before we call the sequence (X_n) of integrable simple functions used in the definition above, a *defining sequence for the integrable function* X.

Proposition

Let X be an integrable function defined on J with values in $\mathcal{L}^2$ with defining sequence (X_n) of integrable simple functions. Then $|X|$ is integrable and for each $S \in \mathcal{B}$ the limits

$$\nu(S) := \lim_{n\to\infty} \int_S X_n \, d\mu_A \text{ and } |\nu|(S) := \lim_{n\to\infty} \int_S |X_n| \, d\mu_A$$

exist in the topological space $\sigma(\mathcal{L}^1, P)$. These limits are independent of the defining sequence and are uniform in S.

Definition

If (X_n) is a defining sequence of integrable simple functions for the integrable function X, we define for all $S \in \mathcal{B}$,

$$\int_S X \, d\mu_A := \lim_{n\to\infty} \int_S X_n \, d\mu_A,$$

with the limit taken in the space $\sigma(\mathcal{L}^1, P)$.

- Lebesgue's theorem as formulated above holds for the general case.

- The notion of progressively measurable can be extended as follows:
- The process $(X(t))$, with X measurable is called *progressively measurable* if for every $t \in [a, b]$, we have that X is measurable on $[a, t]$.
- We denote the set of all Dobrakov integrable functions defined on $J = [a, b]$ by $L^1([a, b], \mu_A)$.

- We denote by $L^2([a,b],\mu_A)$ the space of all μ_A-integrable functions X from $[a,b]$ in $\mathfrak{E}_s$ satisfying

$$\bar{q}_\phi(X)^2 := |\phi|\mathbb{F}\int_a^b |X|^2\, d\mu_A < \infty \text{ for all } \phi \in \mathfrak{E}_{00}^{\sim}.$$

- The set of all these semi-norms is denoted by $\bar{Q}$.
- It is easy to check that $\langle X, Y\rangle := |\phi|\mathbb{F}\int_a^b XY\, d\mu_A$ is a bilinear form and that the Cauchy inequality holds:

$$\left| |\phi|\mathbb{F}\int_a^b XY\, d\mu_A \right| \le \bar{q}_\phi(X)\bar{q}_\phi(Y).$$

Natural processes

Definition

A right-continuous increasing process $(A_t)_{t\in J}$ satisfying

$$\phi\mathbb{F}(M_tA_t) = \phi\mathbb{F}\int_a^t M_s\,d\mu_A = \phi\mathbb{F}\int_a^t M_{s-}d\mu_A \text{ for all } \phi \in \mathfrak{E}_{00}^{\sim}$$

and for all bounded martingales (M_t) is called a *natural process*.

- The notion of a natural process was the instrument used to prove uniqueness of the Doob-Meyer decomposition (see [33]) of submartingales in the classical case.
- This is also the case in the abstract setting with the definition given above.

Definition

Let $(X_t, \mathbb{F}_t)_{t\in J}$ be a submartingale adapted to the filtration $(\mathbb{F}_t)_{t\in J}$. A *Doob-Meyer decomposition* of X_t is a decomposition

$$X_t = M_t + A_t$$

where (M_t) is a martingale and (A_t) is a right-continuous increasing process.

Proposition

Let $X = (X_t, \mathbb{F}_t)_{t\in J}$ *be a submartingale adapted to the filtration* $(\mathbb{F}_t)_{t\in J}$. *Then* X_t *admits only one Doob-Meyer decomposition with natural* (A_t).

The stochastic integral with reference to a martingale

Let $(M_t, \mathfrak{F}_t)$ be a right-continuous martingale with respect to the filtration $(\mathfrak{F}_t)$ with left-hand limits.

- We know that the submartingale M_t^2 has a unique Doob-Meyer decomposition $M_t^2 = L_t + A_t$, where L is a martingale and A is a right-continuous, increasing predictable process and $A_a = 0$.
- Use the process A to generate a vector measure in the definition of the Dobrakov integral to construct the Itô integral.

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