Eventual Positivity of Operator Semigroups

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Joint work with Daniel Daners (University of Sydney) and James B. Kennedy (University of Lisbon)

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(d) *And many more...*

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Observation

Nobody has combined these two approaches, yet.

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(b) *individually eventually positive* if, for all $x \in E_+$, the inequality

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$$T^n f \to \int_0^1 f(x)\,dx \cdot \mathbb{1} \quad (n \to \infty).$$

If $f \geq 0$, then $T^n f \geq 0$ for all large n, **but**: this might happen very late if $\int_0^1 f(x)\,dx$ is small compared to $\|f\|_\infty$.

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(a) A (subtle) resolvent estimate.
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(c) Associate a *positive* operator S to the operator T by means of an ultra power argument. □

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(b) *The spectral bound $s(A)$ is a dominant spectral value of A; moreover, $\ker(s(A) - A)$ is spanned by a vector $v \gg_u 0$ and $\ker(s(A) - A')$ contains a strictly positive functional.*

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That's all certainly nice – but is it useful?

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Proof.

It follows from work of Grunau and Sweers [GS98] that $-\Delta^2$ (with the given boundary conditions) fulfils the spectral condition (b) in the above theorem. □

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Let Δ denote the Laplace operator with the above boundary conditions.

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Kreĭn–Rutman type argument $\Rightarrow$ condition (b) in the theorem holds. □

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- Analysis of eventual positivity for Dirichlet-to-Neumann operators on metric graphs.

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- Can one obtain cyclicity results for the spectrum of eventually positive semigroups?
- Develop the perturbation theory of eventually positive semigroups until it reaches a satisfactory state.

The state of the art.

What do we not know, yet? – Open problems:

- Consider the line $s(A) + i\mathbb{R}$. Characterise eventual positivity of $(e^{tA})_{t \geq 0}$ if there exist essential spectral values and/or inifinitely many spectral values on this line.
- Can one obtain cyclicity results for the spectrum of eventually positive semigroups?
- Develop the perturbation theory of eventually positive semigroups until it reaches a satisfactory state.

Your turn!

Daniel Daners.
Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator.
Positivity, 18(2):235–256, 2014.

Daniel Daners and Jochen Glück.
The role of domination and smoothing conditions in the theory of eventually positive semigroups.
Bulletin of the Australian Mathematical Society.
Available online; DOI: 10.1017/S0004972717000260.

Daniel Daners and Jochen Glück.
Towards a perturbation theory for eventually positive semigroups.
To appear in J. Operator Theory.
Preprint available at arxiv.org/abs/1703.10108.

Daniel Daners, Jochen Glück, and James B. Kennedy.
Eventually and asymptotically positive semigroups on Banach lattices.
J. Differential Equations, 261(5):2607–2649, 2016.

Daniel Daners, Jochen Glück, and James B. Kennedy.
Eventually positive semigroups of linear operators.
J. Math. Anal. Appl., 433(2):1561–1593, 2016.

Jochen Glück.
Invariant sets and long time behaviour of operator semigroups.
PhD thesis, Universität Ulm, 2017.
DOI: 10.18725/OPARU-4238.

Jochen Glück.
Towards a Perron-Frobenius theory for eventually positive operators.
J. Math. Anal. Appl., 453(1):317–337, 2017.

Hans-Christoph Grunau and Guido Sweers.
The maximum principle and positive principal eigenfunctions for polyharmonic equations.
In *Reaction diffusion systems (Trieste, 1995)*, volume 194 of *Lecture Notes in Pure and Appl. Math.*, pages 163–182. Dekker, New York, 1998.

F. Shakeri and R. Alizadeh.
Nonnegative and eventually positive matrices.
Linear Algebra Appl., 519:19–26, 2017.

Helmut H. Schaefer.
Banach lattices and positive operators.
Springer-Verlag, New York-Heidelberg, 1974.
Die Grundlehren der mathematischen Wissenschaften, Band 215.