Convergence of positive operator semigroups

joint work with Jochen Glück

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Introduction

Consider a positive semigroup $(T_t)_{t\geq 0}$ on a Banach lattice E, i.e. a familiy of positive operators $T_t \in \mathscr{L}(E)$ such that $T_t T_s = T_{t+s}$. If $T_t x \to x$ as $t \downarrow 0$ for all $x \in E$, $(T_t)_{t\geq 0}$ is a C_0-semigroup.

Aim:
Find sufficient conditions for $(T_t)_{t\geq 0}$ to *converge strongly*, i.e. that

$$\lim_{t\to\infty} T_t x \text{ exists in } E$$

for every $x \in E$.

Greiner's Theorem

Theorem (Greiner 1982)

Let $E = L^p(\Omega, \mu)$, (Ω, μ) σ-finite, $1 \leq p < \infty$.
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Here: $T \in \mathscr{L}(L^p)$ is a *kernel operator* if there exists a measurable $k \colon \Omega \times \Omega \to \mathbb{R}$ such that for every $f \in L^p$

$$(Tf)(x) = \int_\Omega k(x, y) f(y) \, \mathrm{d}\mu(y) \text{ almost everywhere.}$$

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More generally: On an order complete Banach lattice E, $T \in \mathscr{L}(E)$ is a *kernel operator* if $T \in (E' \otimes E)^{\perp\perp}$.

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Theorem (Greiner extended)

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Same problem with dual semigroups on non-reflexive spaces, Gaussian semigroup on $C_b(\mathbb{R})$, …

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How (not) to prove this?

First attempt

Theorem (Lotz, Doob)

For a bounded semigroup $(T_t)_{t\geq 0}$ *on a Banach space* X

$$\lim_{t\to\infty} T_t \text{ exists strongly } \Leftrightarrow \lim_{n\to\infty} (T_t)^n \text{ exists strongly for each } t > 0$$

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$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a positive and contractive kernel operator on $\mathbb{R}^2$ with fixed point $(1,1)$ but $T^n(1,0)$ does not converge!

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$\rightsquigarrow$ This operator has no positive root!

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Example (refinement)

Let $T, S \in \mathbb{R}^{3\times 3}$ be permutation matrices for the cycles $(1,2,3)$ and $(1,3,2)$. Then $T^2 = S$ and $S^2 = T$.

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One has to exploit that every operator has arbitrary positive roots!

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- $\langle S \rangle$ is *divisible*, i.e. $\forall g \in \langle S \rangle \; \forall n \in \mathbb{N} \; \exists h \in \langle S \rangle$ with $nh = g$.

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$(T_t)_{t \in S}$ is *strongly convergent* if $(T_t x)_{t \in S}$ converges for all $x \in E$ with respect to $t \leq s :\Leftrightarrow t = s$ or $\exists r \in S$ such that $s = t + r$

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- The existence of quasi-interior fixed point is cruical!

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- Let $\{v_j : j \in J\}$ basis of $\mathbb{R}$ over $\mathbb{Q}$.
- Fix $i \in J$ and let $V \coloneqq \mathrm{span}\{v_j : j \neq i\}$.

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- Let $\{v_j : j \in J\}$ basis of $\mathbb{R}$ over $\mathbb{Q}$.
- Fix $i \in J$ and let $V := \operatorname{span}\{v_j : j \neq i\}$.
- $V \triangleleft \mathbb{R}$, $\varphi\colon \mathbb{R} \to \mathbb{R}/V \cong \mathbb{Q}$ surjective group homomorphism.

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- $V \triangleleft \mathbb{R}$, $\varphi\colon \mathbb{R} \to \mathbb{R}/V \cong \mathbb{Q}$ surjective group homomorphism.
- The Koopman group $T_t(x_q)_{q\in\mathbb{Q}} := (x_{q+\varphi(t)})_{q\in\mathbb{Q}}$ is positive, contractive but not strongly convergent.

Further generalization

Theorem (Pichór, Rudnicki, 2000)

A Markovian C_0-semigroup $(T_t)_{t\geq 0}$ on L^1 is strongly convergent if

- $(T_t)_{t\geq 0}$ *is irreducible and possesses a quasi-interior fixed point*
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  (we say: T_s is a partial kernel operator).

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- *For each fixed point $x > 0$ there exists $s \in S$ and kernel operator $K_s \geq 0$ such that $T_s \geq K_s$ and $K_s x > 0$.*

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Let $T \colon L^p(\Omega) \to L^p(\Omega)$ positive, Ω second countable topological space with a regular Borel measure.
If $Tf > 0$ is lower semi-continuous for all $f > 0$, then T is a partial kernel operator.