Positivity IX

Edmonton, Canada, July 17, 2017

Higher order Jensen-convex functionals that are measurable on curves

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Some eighty years ago Stanisław Mazur asked the following question (see "Problem 24" from the famous "Scottish Book" [6]):

> ***In a space*** E ***of type*** (B), ***there is given an additive functional*** $F(x)$ ***with the following property: If*** $x(t)$ ***is a continuous function in*** $0 \leqslant t \leqslant 1$ ***with values in*** E, ***then*** $F(x(t))$ ***is a measurable function. Is*** $F(x)$ ***continuous ?***

Half century later, the solution, in the affirmative, was given by I. Labuda and R. D. Mauldin [4]. As a matter of fact, they have proved a more general theorem: instead of functionals they considered additive operators from a Banach space into a Hausdorff topological vector space. Fairly soon afterwards this result was generalized by Z. Lipecki [5] to the case where the domain and range of the additive transformations considered are suitable Abelian topological groups.

111 years ago Danish mathematician

J.L.W.V. Jensen

(*Sur les fonctions convexes et les inégualités entre les valeurs moyennes*, Acta Math. 30 (1906), 175-193) expressed the following opinion:

"Il me semble que la notion de fonction convexe est à peu près aussi fondamentale que celles-ci fonction positive, fonction croissante. Si je ne me trompe pas en ceci, la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles.

And he was certainly not mistaken!

It is widely known (see e.g. M. Kuczma [3]) that the regularity behaviour of the so called Jensen-convex functionals is, in general, very similar to that of additive ones. Therefore, it seems natural to ask after the Jensen-convex analogue of Mazur's problem.

In [1] the following result has been proved.

THEOREM 1. ***Let*** $(E, \|\cdot\|)$ ***be a real Banach space and let*** D ***be a nonempty open and convex subdomain of*** E. ***Then each Jensen-convex functional*** $F : D \longrightarrow \mathbb{R}$ ***having the property that the superposition*** $F \circ x$ ***is Lebesgue measurable for every continuous map*** $x : [0, 1] \longrightarrow D$, ***is continuous (and hence also convex).***

For *finite dimensional* spaces this result may essentially be strengthened. Namely we have the following

THEOREM 2. ***Let*** $(E, \|\cdot\|)$ ***be a real Banach space and let*** D ***be a nonempty open and convex subdomain of*** E. ***Then each Jensen-convex functional*** $F : D \longrightarrow \mathbb{R}$ ***having the property that the superposition*** $F \circ x$ ***is Lebesgue measurable for every affine map*** $x : [0, 1] \longrightarrow D$, ***is convex. In particular, if the space*** $(E, \|\cdot\|)$ ***is finite dimensional, then*** F ***is also continuous.***

What about convexity of higher orders? More exactly, the question reads as follows:

Given a nonempty open and convex subdomain D of a real Banach space $(E, \|\cdot\|)$ endowed with a cone C of positive elements with $C \cap (-C) = \{0\}$, assume that F is an n-th order Jensen-convex functional on D, i.e. F is a solution to the conditional functional inequality

$$v-u \in C \Longrightarrow \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left((1-\frac{j}{n+1})u + \frac{j}{n+1}v\right) \geqslant 0\,,$$

valid for all $u, v \in D\,.$

Does the requirement that the superposition $F \circ x$ is Lebesgue measurable for every continuous map $x : [0,1] \longrightarrow D$, force F to be continuous ? To give a partial answer to that question we need to recall some facts.

1. **Any continuous (first order) Jensen-convex functional $F : D \longrightarrow \mathbb{R}$ is convex in the usual sense, i.e. it satisfies the inequality**

$$F(\lambda x + (1-\lambda)y) \leqslant \lambda F(x) + (1-\lambda)F(y)$$

for all vectors x, y from D and all scalars λ from the unit interval $[0,1]$.

2. Any continuous n-th order Jensen-convex functional $F : D \longrightarrow \mathbb{R}$ is n-convex in the sense of Tiberiu Popoviciu (see e.g. M. Kuczma [3]), i.e. it satisfies the conditional inequality

$$v - u \in C \Longrightarrow$$

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_{n+1}) F((1-\lambda_j)u+\lambda_j v) \geqslant 0$$

for every choice of real numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$, where V stands for the Vandermonde's determinant of the variables considered (a higher order counterpart of the standard convexity and reducing to it in the case $n = 1$).

3. The linear structure of a real linear space E induces the so called *core*-topology in E as follows: given a set $G \subset E$ denote by core G the set of all points $y \in G$ enjoying the property that for every $x \in E$ one may find an $\varepsilon > 0$ such that $y + tx \in G$ for all $t \in (-\varepsilon, \varepsilon)$. Then G is said to be algebraically open provided that $G = \operatorname{core} G$. The family of all algebraically open sets forms a topology in E which is just termed *core*-topology. Unfortunately, in general, a linear space with its *core*-topology fails to be a topological linear space: instead of having the joint continuity of the map

$$E \times E \times \mathbb{R} \ni (x, y, \lambda) \longmapsto \lambda x + y \in E$$

we get merely the separate continuity of it (semilinearity). The *core*-topology of a linear space E turns out to be the finest possible semilinear topology in E. There exist numerous sources in mathematical literature where these topologies are studied. In what follows the knowledge of Chapter I in Z. Kominek's dissertation [2] is utterly sufficient.

Theorem 3. ***Let $(E, \|\cdot\|), D$ and C have the meaning described above. Then any solution $F : D \longrightarrow \mathbb{R}$ of the conditional functional inequality***

$$v-u \in C \Longrightarrow \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}F\left((1-\frac{j}{n+1})u+\frac{j}{n+1}v\right) \geqslant 0\,,$$

for all $u, v \in D$, having the property that the superposition $F \circ x$ is Lebesgue measurable for every affine map $x : [0,1] \longrightarrow D$, is $n-$convex. In the case where the order relation generated by the cone C in question is linear (i.e. $C \cup (-C) = E$) F is also continuous in the core-topology of E.

Remark 1. **Noteworthy is the fact that the family of all *continuous* "testing functions" $x : [0, 1] \longrightarrow D$ used in the statement of Mazur's problem has been restricted *merely* to affine functions.**

Remark 2. **The question whether or not F is continuous in the *norm* topology of the space $(E, \|\cdot\|)$ considered remains open even under the Mazur's assumption that the superposition $F \circ x$ is Lebesque measurable for *every* continuous map $x : [0, 1] \longrightarrow E$.**

However, since in finite dimensional spaces $n-$convex functions on open and convex domains are automatically continuous (see e.g. M. Kuczma **[3]), a positive answer to the corresponding Mazur's question results immediately from Theorem 1. Namely, we have the following:**

Theorem 4. ***Given a nonempty open and convex subdomain D of a finite dimensional real Banach space endowed with a cone C of positive elements with $C \cap (-C) = \{0\}$, assume that F is an $n-$th order Jensen-convex functional on D, i.e. F is a solution to the conditional functional inequality***

$$v-u \in C \Longrightarrow \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F\left((1 - \frac{j}{n+1})u + \frac{j}{n+1} v\right) \geqslant 0$$

valid for all $u, v \in D$. Suppose that the superposition $F \circ x$ is Lebesgue measurable for every affine map $x : [0, 1] \longrightarrow D$. Then F is continuous.

Remark 3. **Without any changes in the proof one mi-ght replace the assumption that**

the superposition $F \circ x$ ***is Lebesgue measurable for every affine map*** $x : [0, 1] \longrightarrow D$

by the requirement that

for every affine map $x : [0, 1] \longrightarrow D$, ***the superposition*** $F \circ x$ ***is bounded on a second category Baire subset of*** D***, that may depend upon*** x***,***

or any other alternative assumption forcing an n**-th order Jensen-convex functional on an open subinterval of the real axis to be continuous. Numerous such sufficient conditions may be found in** M. Kuczma's **monograph [3], for instance.**

Theorem 5. ***Let*** $(E, \|\cdot\|)$ ***be a real Banach space and let*** D ***be a nonempty open and convex subdomain of*** E. ***Let further*** $F : D \longrightarrow \mathbb{R}$ ***be a Wright-convex functional, i.e.*** F ***solves the functional inequality***

$$F(\lambda x + (1-\lambda)y) + F(\lambda y + (1-\lambda)x) \leqslant F(x) + F(y)$$

for all $x, y \in D$. ***If the superposition*** $F \circ x$ ***is Lebesgue measurable for every affine map*** $x : [0,1] \longrightarrow D$, ***then*** F ***is convex. In particular, if the space*** $(E, \|\cdot\|)$ ***is finite dimensional, then*** F ***is also continuous.***

R e f e r e n c e s

[1] R. GER, Mazur's criterion for continuity of convex functionals, *Bulletin of the Polish Academy of Sciences, Mathematics* 43 (1995), 263-268.

[2] Z. KOMINEK, Convex functions in linear spaces, *Prace Naukowe Uniwersytetu Śląskiego w Katowicach* 1087*, Katowice*, 1989.

[3] M. KUCZMA, An introduction to the theory of functional equations and inequalities, *Polish Scientific Publishers & Uniwersytet Śląski, Warszawa-Kraków-Katowice*, 1985 (second edition: *Birkhäuser Verlag, Basel-Boston-Berlin*, 2009).

[4] I. LABUDA and R. D. MAULDIN, Problem 24 of the "Scottish Book" concerning additive functionals, *Colloquium Mathematicum* 48 (1984), 89-91.

[5] Z. LIPECKI, On continuity of group homomorphisms, *Colloquium Mathematicum.* 48 (1984), 93-94.

[6] The Scottish Book, Mathematics from the Scottish Café, *Edited by R. Daniel Mauldin, Birkhäuser Verlag, Boston-Basel-Stuttgart*, 1981.