On weakly p-summable sequences in Banach lattices

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Recall the following result:

Theorem 0.1 *(See for instance Theorem 4.34 in [3]) If W is a relatively weakly compact subset of a Banach lattice, then every disjoint sequence in the solid hull of W converges weakly to zero.*

This theorem plays an important role in the proofs of many results (for instance, concerning Dunford-Pettis operators and the Dunford-Pettis property on Banach lattices).

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[0] [3] Aliprantis, C. D., Burkinshaw, O. "Positive Operators".

The objective in this talk is to introduce the notion of “$(1, p)$-limited” subset of a Banach space and to show that:

”If E is a Banach lattice and $(x_n) \subset E$ is a disjoint sequence in the solid hull of a $(1, p)$-limited subset of E, then (x_n) is weakly p-summable in E”

– and then to consider some applications of this result to operators on Banach lattices.

1 Notation

For Banach spaces X, Y, we let:

$\mathcal{L}(X,Y)$ be the space of bounded linear operators and $\mathcal{L}(X,\mathbb{K}) = X^*$;

$\mathcal{K}(X,Y)$ be the space of compact linear operators;

$\mathcal{W}(X,Y)$ be the space of weakly compact linear operators;

$\ell_p^{strong}(X)$ $(1 \leq p < \infty)$ be the space of all p-summable sequences in X, i.e

$$(x_n) \in \ell_p^{strong}(X) \iff (\|x_n\|) \in \ell_p;$$

$\ell_p^{weak}(X)$ $(1 \leq p < \infty)$ be the space of all weakly p-summable sequences in X, i.e.

$(x_n) \in \ell_p^{weak}(X) \iff (\langle x^*, x_n\rangle) \in \ell_p$, $\forall x^* \in X^*$;

$\ell_p^u(X)$ be the closure in $\ell_p^{weak}(X)$ of the set of all finitely non-zero sequences in X;

$\ell_p^{weak^*}(X^*)$ $(1 \leq p < \infty)$ be the space of all weak* p-summable sequences in X, i.e.

$(x_n^*) \in \ell_p^{weak^*}(X^*) \iff (\langle x_n^*, x\rangle) \in \ell_p$, $\forall x \in X$

(note that $\ell_p^{weak^*}(X^*) = \ell_p^{weak}(X^*)$)

$c_0^{weak}(X)$ be the vector space of all weakly null sequences in X;

Remark 1.1

(1) The elements of $\ell_p^u(X)$ are called the unconditionally p-summable sequences in X. It is well-known that $(x_n) \in \ell_1^u(X)$ if and only if (x_n) is an unconditionally summable sequence in X and $\ell_1^{weak}(X) = \ell_1^u(X)$ if and only if X does not contain a copy of c_0.

(2) All Banach lattices will be assumed to be real and will be denoted by E, F, G etc.

2 Weakly p-summable sequences in Banach lattices

We assume throughout this section that $1 \leq p < \infty$.

Remark 2.1 *Suppose E is a Banach lattice and $(x_n) \in \ell_p^{weak}(E)$ satisfies $x_n \geq 0$ for all n. Suppose $y_n \in E$ satisfies $0 \leq y_n \leq x_n$ for all n. One verifies readily that $(y_n) \in \ell_p^{weak}(E)$ as well.*

In general, $(x_n) \in \ell_p^{weak}(E)$ does not necessarily imply $(|x_n|) \in \ell_p^{weak}(E)$. However, a standard argument argument yields that:

Proposition 2.2 *Suppose (x_n) is a disjoint sequence in E. Then,*

$$(x_n) \in \ell_p^{weak}(E) \iff (|x_n|) \in \ell_p^{weak}(E).$$

It follows from Proposition 2.2 and $x_n^+ \leq |x_n|$ and $x_n^- \leq |x_n|$ for all n, that:

Corollary 2.3 *Let E be a Banach lattice and $(x_n) \subset E$ a disjoint sequence. Then,*

$$(x_n) \in \ell_p^{weak}(E) \iff (x_n^+), (x_n^-) \in \ell_p^{weak}(E).$$

We recall the notion of "weak p-consistent":

Definition 2.4 *We say a Banach lattice E is weak p-consistent (for $1 \leq p < \infty$) if it follows from $(x_n) \in \ell_p^{weak}(E)$ that $(|x_n|) \in \ell_p^{weak}(E)$.*

It is known that:

Lemma 2.5 *Let $1 \leq p < \infty$. The space $C(\Omega)$ is weak p-consistent and therefore, all AM-spaces with unit are weak p-consistent.*

Recall the following definition of a p-limited subset of a Banach space:

Definition 2.6 *(refer to [9]) Let* $1 \leq p < \infty$*. A subset* W *of a Banach space* X *is said to be* p*-limited if for each weak** p*-summable sequence* (x_n^*) *in* X^* *there exists a sequence* $(\lambda_i) \in \ell_p$ *such that*

$$|\langle x_n^*, x\rangle| \leq \lambda_n, \quad \forall x \in W,$$

and for each $n \in \mathbb{N}$*.*

If in the above definition we allow $p = \infty$ (and then replace ℓ_∞ by c_0), then we have the well-known definition of a *limited set*.

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[0] [9] Delgado, J.M.; Piñeiro, C. "On p-limited sets". J.Math.Anal.Appl.(2014)

It follows from [9] that every p-limited set is relatively weakly compact and that if $1 \leq p \leq q < \infty$, then every p-limited set is q-limited.

However, it is not necessarily true that a p-limited set is limited (for instance, B_{ℓ_2} is 1-limited in c_0, but not limited).

If X is a Grothendieck space (i.e. weak* convergent sequences in X^* are weakly convergent) then each p-limited set is indeed limited.

If, however, $2 \leq p < q < \infty$ and every q-limited subset of X is p-limited, then X has to be finite dimensional.

A Banach space X is said to have the **Gelfand-Phillips property** (GPP for short) or X is said to be a Gelfand-Phillips space, if all limited subsets of X are relatively (norm) compact. This is the case if and only if every limited weakly null sequence in X is norm null.

Similarly,

[0] [9] Delgado, J.M.; Piñeiro, C. "On p-limited sets". J.Math.Anal.Appl.(2014)

Definition 2.7 *Let* $1 \leq p < \infty$. *A Banach space* X *is said to have the p-Gelfand-Phillips property (pGPP for short) if every limited weakly p-summable sequence* (x_n) *in* X *is norm null. If* X *has this property, then we call* X *a p-Gelfand-Phillips space.*

The Definition 2.6 above (of p-limited set), extends to:

Definition 2.8 *A subset* W *of a Banach space* X *is said to be* (p, q)*-limited (where* $1 \leq p, q < \infty$*) if for each weak* p-summable sequence* (x_n^*) *in* X^* *there exists a sequence* $(\lambda_i) \in \ell_q$ *such that*

$$|\langle x_n^*, x \rangle| \leq \lambda_n, \quad \forall x \in W,$$

and for each $n \in \mathbb{N}$.

It is immediate that the (p, p)-limited sets are the p-limited sets. Clearly, if $1 \leq r \leq p$, then each (p, q)-limited set is (r, q)-limited, i.e. each (p, q)-limited set is $(1, q)$-limited. On the other hand, if $1 \leq r \leq q$, then each (p, r)-limited set is (p, q)-limited.

Again, we may include $p = \infty$ or/and $q = \infty$ in Definition 2.8, if ℓ_∞ is replaced by c_0. Thus, a set $A \subset X$ is (p, ∞)-limited if for each weak* p-summable sequence (x_n^*) in X^* there exists $(\lambda_n) \in c_0$ such that

$$\sup_{x \in A} |\langle x_n^*, x\rangle| \leq \lambda_n, \ \forall n \in \mathbb{N}.$$

Let A be a weakly compact subset of X. For each $(x_n^*) \in \ell_1^{weak}(X^*)$, the set $\{(\langle x_n^*, x\rangle)_n : x \in A\}$ is weakly compact in ℓ_1 as image of A under the bounded linear operator

$$X \to \ell_1 : x \mapsto (\langle x_n^*, x\rangle),$$

and so it is compact in ℓ_1. Thus it is contained in the closed convex hull of a norm null sequence $((\lambda_{ni})_i)_n$ in ℓ_1. It then follows that

$$\sup_{x \in A} |\langle x_n^*, x\rangle| \leq \|(\lambda_{ni})_i\|_{\ell_1} \text{ for all } n,$$

where $\|(\lambda_{ni})_i\|_{\ell_1} \to 0$ as $n \to \infty$. This shows that:

Remark 2.9 *All weakly compact sets in a Banach space X are $(1, \infty)$-limited.*

Recall that:

Definition 2.10 *(Type and cotype) A Banach space has type p $(1 \le p \le 2)$ if there exists a constant $C \ge 0$ such that, however we choose finitely many $x_1, \ldots, x_n$ from X,*

$$\left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 \, dt \right)^{1/2} \le C (\sum_{k=1}^n \|x_k\|^p)^{1/p}$$

and it has cotype q $(2 \le q \le \infty)$ if there is a constant $K \ge 0$ such that no matter how we select finitely many $x_1, \ldots, x_n$ from X,

$$(\sum_{k=1}^n \|x_k\|^q)^{1/q} \le K \left(\int_0^1 \| \sum_{k=1}^n r_k(t) x_k \|^2 \, dt \right)^{1/2},$$

where $r_n : [0,1] \to \mathbb{R}$ denotes the Rademacher function $r_n(t) := \operatorname{sign}(\sin 2^n \pi t)$ and where $q = \infty$ is covered by replacing the left hand side by $\max_{k \le n} \|x_k\|$.

Next we discuss some examples of Banach spaces in which all bounded sets are $(1,p)$-limited for some p:

Example 2.11

1. *Clearly,* B_X *in* X *is* $(1,p)$*-limited iff*
$$(\dagger) \qquad \ell_1^{weak}(E^*) \subseteq \ell_p^{strong}(E^*).$$
If a Banach space X *has type* $1 < p' \leq 2$*, then* X^* *has cotype* p *and* $(\dagger)$ *holds (see [11], Theorem 11.17). Thus, all bounded sets in a Banach space* X *with type* $1 < p' \leq 2$ *are* $(1,p)$*-limited. In case of a Banach lattice* E *with finite cotype, then* E *has type* p' *if and only if* E^* *has cotype* p *and this is so, if and only if* $\ell_1^{weak}(E^*) \subseteq \ell_p^{strong}(E^*)$.

2. *The assumption that* X *has non-trivial type in the previous example is not necessary. For instance, the space* ℓ_1 *has cotype 2, which implies that the inclusion* $\ell_1^{weak}(\ell_1) \subseteq \ell_2^{strong}(\ell_1)$ *holds; i.e.* $\ell_1^{weak}(c_0^*) \subseteq \ell_2^{strong}(c_0^*)$*. Thus the bounded sets in* c_0 *are* $(1,2)$*-limited. Recall that* c_0 *does not have non-trivial type.*

   Similarly, since $C(K)^* = \ell_1$ *for every countable compact metric space* K*, it follows that the bounded sets in* $C(K)$ *are* $(1,2)$ *limited.*

[0] [11] Diestel, J., Jarchow, H., Tonge, A.: Absolutely summing operators

3. *Recall that a Banach space X is said to have Orlicz property, when unconditionally convergent series in X are strongly 2-summable. If X does not contain a copy of c_0, then it satisfies the Orlicz property if and only if*

$$\ell_1^{weak}(X) \subseteq \ell_2^{strong}(X).$$

   Therefore, if X is a Banach space such that its dual space X^* does not contain a copy of c_0 and satisfies the Orlicz property, then all bounded sets in X are $(1,2)$-limited.

   The previous example where $X = c_0$ is a special case.

   Let $K = [0,1]$. The space $C(K)$ fails to have non-trivial type, whereas its dual space $\mathcal{M}(K)$ (of finite regular Borel measures on K) has cotype 2. Therefore, $\mathcal{M}(K)$ satisfies the Orlicz property, i.e. the bounded sets in $C(K)$ are $(1,2)$ limited.

The proof of the following theorem is based on the proof of Theorem 0.1 (Theorem 4.34 in [3]):

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[0] [3] Aliprantis, C. D., Burkinshaw, O. "Positive Operators".

Theorem 2.12 *Let $1 \leq p \leq \infty$. Suppose $(x_n) \subset E$ is a disjoint sequence in the solid hull of a $(1,p)$-limited set W. Then, $(x_n) \in \ell_p^{weak}(E)$.*

Proof Let (x_n) be the sequence in the statement of the theorem. Pick a sequence $(y_n) \subseteq W$ satisfying $|x_n| \leq |y_n|$ for all n. Fix $0 \leq x^* \in E^*$.

Considering each x_n as an element of E^{**}, denote by P_n the order projection of E^* onto the carrier C_{x_n} of x_n. Using that $x_n \perp x_m$ $(n \neq m)$, and so (by Nakano) $P_n x^* \perp P_m x^*$ holds for $n \neq m$, it follows for the given positive functional x^* that

$$\begin{aligned} |x^*(x_n)| \leq x^*(|x_n|) &= [P_n x^*](|x_n|) \\ &\leq [P_n x^*](|y_n|) \\ &= \max\{y^*(y_n) : |y^*| \leq P_n x^*\}, \end{aligned}$$

for each $n \in \mathbb{N}$. Thus, for each n, there exists some $y_n^* \in E^*$ with $|y_n^*| \leq P_n x^*$ and

$$|x^*(x_n)| \leq y_n^*(y_n). \qquad (*)$$

For each $x \in E$ and each $k \in \mathbb{N}$, we have

$$\sum_{i=1}^{k} |y_i^*(x)| \leq \left[\sum_{i=1}^{k} P_i x^*\right](|x|) \leq x^*(|x|),$$

and so $(y_i^*) \in \ell_1^{weak^*}(E^*)$. Since W is a $(1,p)$-limited set, it follows that there exists a sequence $(\lambda_n) \in \ell_p$ so that

$$|x^*(x_n)| \leq y_n^*(y_n) \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

This shows that $(x^*(x_n)) \in \ell_p$ for all $0 \leq x^* \in E^*$, from which it follows that for all $x^* \in E^*$ and for all $(\alpha_i) \in \ell_{p'}$, we have

$$\sum_{n=1}^{\infty} |\alpha_n x^*(x_n)| = \sum_{n=1}^{\infty} |\alpha_n||x^*(x_n)| \leq$$
$$\sum_{n=1}^{\infty} |\alpha_n||(x^*)^+(x_n)| + \sum_{n=1}^{\infty} |\alpha_n||(x^*)^-(x_n)| < \infty,$$

thereby showing that $(x_i) \in \ell_p^{weak}(E)$. $\square$

Theorem 0.1 follows from theorem2.12, since relatively weakly compact sets are $(1,\infty)$-limited.

3 Applications to classes of operators on Banach lattices

We recall the following definitions from [6]:

[0] [6] Castillo, J.M.F., Sánchez, F. "Dunford-Pettis-like properties of continuous vector function spaces", *Revista Mat.Complut.Madrid* **6(1)**(1993)

Definition 3.1

(1.) A sequence (x_n) in a Banach space X is called weakly p-convergent if there exists $x \in X$ such that $(x_n - x) \in \ell_p^{weak}(X)$. A subset W of a Banach space X is called relatively weakly p-compact if each sequence $(x_n) \subseteq W$ has a weakly p-convergent subsequence. If a relatively weakly p-compact set contains the "limits" of all its weakly p-convergent sequences, then it is called weakly p-compact.

(2.) An operator $T : X \to Y$ is called p-convergent if $\|Tx_n\| \to 0$ for all $(x_n) \in \ell_p^{weak}(X)$.

We recall the well-known Dunford-Pettis property and similar properties that were studied widely in the literature in recent years:

Definition 3.2 *A Banach space X is said to have*

(1.) DPP (Dunford-Pettis property), if for all Banach spaces Y each weakly compact operator $T : X \to Y$ maps weakly compact sets to norm-compact sets (i.e. T is Dunford-Pettis or completely continuous) or, equivalently, if $x_n^* \to 0$ weakly in X^* and $x_n \to 0$ weakly in X imply $x_n^*(x_n) \to 0$;

(2.) DP^*P, if all weakly compact subsets of X are limited (equivalently, if each bounded linear operator $T : X \to c_0$ is completely continuous or, equivalently, if $x_n^* \to 0$ weak in X^* and $x_n \to 0$ weakly in X, then $x_n^*(x_n) \to 0$);*

(3.) DPP_p (Dunford-Pettis property of order p), if each weakly compact operator

$$T : X \to Y$$

is p-convergent or, equivalently, if $x_n^* \to 0$ weakly in X^* and $(x_n) \in \ell_p^{weak}(X)$ imply

$$x_n^*(x_n) \to 0.$$

(4.) DP^*P_p (DP^* of order p), if all weakly p-compact sets in X are limited (equivalently, if each bounded linear operator $T : X \to c_0$ is p-convergent or, equivalently, if $x_n^* \to 0$ weak in X^* and $(x_n) \in \ell_p^{weak}(X)$ imply $x_n^*(x_n) \to 0$)*

In the light of the above discussion, we have:

Definition 3.3 *Let $p \leq q$. We say a Banach space X has the $DP^*P_{(p,q)}$ if each weakly q-compact subset of X is (p, q)-limited.*

Refer to Example 2.11 above for examples of Banach spaces with $DP^*P_{(1,p)}$ (for some values of p).

The well-known Kalton-Saab Theorem states:

Theorem 3.4 *Let E, F be Banach lattices such that F has order continuous norm. If a positive operator $S : E \to F$ is dominated by a Dunford-Pettis operator, then S itself is Dunford-Pettis.*

Using our Theorem 2.12, the proof of Theorem 3.4 (as is discussed in [3]) can be adjusted to show that:

Theorem 3.5 *Let E, F be Banach lattices such that E has $DP^*P_{(1,p)}$ (with $1 \leq p < \infty$) and F has order continuous norm. If $T : E \to F$ is a positive p-convergent operator, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is p-convergent itself.*

Aliprantis and Burkinshaw introduced the class of weak Dunford-Pettis operators. Recall that an operator $T : X \to Y$ is weak Dunford-Pettis if it follows from $x_n \to 0$ weakly in X and $y_n^* \to 0$ weakly in Y^*, that $\lim_n \langle y_n^*, Tx_n \rangle \to 0$. Again, due to N.J. Kalton and P. Saab, we have

Theorem 3.6 *If a positive operator S is dominated by a weak Dunford-Pettis operator, then S is a weak Dunford-Pettis operator.*

In our context we consider the weak* p-convergent operators:

Definition 3.7 *An operator T from a Banach space X into a Banach space Y is called weak* p-convergent if $(y_n^*(Tx_n))$ converges to 0 for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak^*}(Y^*)$.*

By a result in the paper [8], the σ-Dedekind completeness of a Banach lattice F assures that that both the sequences of positive parts and absolute values of a disjoint weak* null sequence in F^* are weak* null themselves. Using this result, one proves that

Lemma 3.8 *Let E, F be Banach lattices such that F is σ-Dedekind complete and let $T : E \to F$ be a positive weak* p-convergent operator. Then for every weakly p-summable sequence (x_n) in E^+ and every weak* null sequence (f_n) in F^*, we have*

$$|f_n|(Tx_n) \to 0 \text{ as } n \to \infty.$$

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[0] [8] CHEN,J.X.; CHEN, Z.L. & JI, G.X. "Almost limited sets in Banach lattices", *J.Math.Anal.Appl.* **412** (2014)

Notice that by the definition of weak* p-convergent operator, it follows that for the sequences (x_n) and (f_n) in the statement of Lemma 3.8, we already have $f_n(Tx_n) \to 0$ as $n \to \infty$. The important consequence of the σ-Dedekind completeness of F is that we have the stronger property $|f_n|(Tx_n) \to 0$ as $n \to \infty$.

Remark 3.9 *For $1 < p < \infty$, T is weak* p-convergent if and only if it carries relatively weakly p-compact subsets of X to limited subsets of Y.*

Based on the discussion by Kalton and Saab in [18] (to prove Theorem 3.6) and Lemma 3.8 we verify that:

Theorem 3.10 *Let $T : E \to F$ be a positive weak* p-convergent operator (for $1 \leq p < \infty$), where E, F are Banach lattices such that E is weak p-consistent and F is σ-Dedekind complete. If $0 \leq S \leq T$, then S is weak* p-convergent.*

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[0] [18] KALTON, N.J. & SAAB,P. "Ideal properties of regular operators between Banach lattices", *Illinois J.Math.* **29** (1985)

Proof Let $(x_n) \in \ell_p^{weak}(E)$ and let $(f_n) \in c_0^{weak^*}(F^*)$. By assumption, $(|x_n|) \in \ell_p^{weak}(E)$ and $T : E \to F$ is weak* p-convergent. So, by Lemma 3.8, we have

$$|f_n|(T|x_n|) \to 0 \text{ as } n \to \infty.$$

This implies that $f_n^+(T|x_n|) \to 0$ and $f_n^-(T|x_n|) \to 0$ as $n \to \infty$. Therefore,

$$|f_n^+(Sx_n)| \leq f_n^+(|Sx_n|) \leq f_n^+(S|x_n|) \leq f_n^+(T|x_n|) \to 0.$$

Similarly, $|f_n^-(Sx_n)| \to 0$ as $n \to \infty$. This proves that $f_n(Sx_n) \to 0$ as $n \to \infty$. $\square$

Corollary 3.11 *Let $T : E \to F$ be a positive weak* p-convergent operator (for $1 \leq p < \infty$), where E, F are Banach lattices such that E is an AM-space with unit and F is σ-Dedekind complete. If $0 \leq S \leq T$, then S is weak* p-convergent.*

By using Theorem 2.12 and following similar arguments to the proof of a theorem in [7] (again, based on results in [3]), we have:

[0] [7] CHEN,J.X., CHEN Z.L. & JI, G.X. "Domination by positive weak* Dunford-Pettis operators on Banach lattices", *Bull. Aust. Math. Soc.* 90 (2014)

Theorem 3.12 *Let $T : E \to F$ be a positive weak* p-convergent operator, where E, F are Banach lattices such that E has $DP^*P_{(1,p)}$ (with $1 \leq p < \infty$) and F is σ-Dedekind complete. Given a weak p-summable sequence (z_n) in E, let W be the set of elements in the sequence (z_n). If $f_n \to 0$ weak* in F^*, then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ and some $w \in E^+$ lying in the ideal generated by W such that*

$$|f_n|(T(|x| - w)^+) < \epsilon,$$

for all $n > N$ and all $x \in W$.

Application of Theorem 3.12 then yields the following extension of Theorem 3.10:

Theorem 3.13 *Let E, F be Banach lattices such that E has $DP^*P_{(1,p)}$ (with $1 \leq p < \infty$) and F is σ-Dedekind complete. If $T : E \to F$ is a positive weak* p-convergent operator, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is weak* p-convergent itself.*

The following proposition provides a connection between the weak* p-convergent and p-convergent operators:

Proposition 3.14 *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. The following are equivalent:*

(a) T is weak p-convergent.*

(b) ST is p-convergent for each $S \in \mathcal{L}(Y, Z)$ and any separable Banach space Z.

(c) ST is p-convergent for each $S \in \mathcal{L}(Y, c_0)$.

It follows from Proposition 3.14 that

Corollary 3.15 *If X, Y are Banach spaces, with Y separable, then each weak* p-convergent operator $T : X \to Y$ is p-convergent.*

Corollary 3.16 *A Banach space X has DP^*P_p if and only if the identity operator id_X is weak* p-convergent. If X is separable, then by Corollary 3.15, this is equivalent to id_X being p-convergent.*

4 The Schur and positive Schur properties of order p

If the lattice operations in a Banach lattice E are weakly sequentially continuous, then in particular $|x_n| \to 0$ weakly for all $(x_n) \in \ell_p^{weak}(E)$. It is well-known that the lattice operations in AM-spaces are weakly sequentially continuous. However, in the spaces $L_p[0,1]$ (where $1 \leq p < \infty$) the lattice operations fail to be weakly sequentially continuous (see [19]). It is also proved in [19] that in every atomic Banach lattice with order continuous norm, the lattice operations are weakly sequentially continuous. Since we need the lattice operations to satisfy a seemingly weaker property than being weakly sequentially continuous, we introduce the notion "weakly sequentially p-continuous" as follows:

Definition 4.1 *The lattice operations in a Banach lattice E are said to be weakly sequentially p-continuous if the sequence $(|x_n|)$ converges weakly to 0 for every weakly p-summable sequence (x_n).*

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[0] [19] Meyer-Nieberg, P. "Banach lattices", Springer-Verlag, Berlin, Heidelberg, 1991

Definition 4.2 *Let* $1 \le p < \infty$. *A Banach lattice* E *is said to have the positive Schur property of order* p *(briefly,* E *has the* SP_p^+*) if each sequence* $(x_n) \in \ell_p^{weak}(E)$ *with positive terms, is norm convergent to 0.*

If we agree to say that E has the SP_∞^+ if each sequence $(x_n) \in c_0^{weak}(E)$ with positive terms, is norm convergent to 0, then we may assume $1 \le p \le \infty$ in Definition 4.2; the SP_∞^+, however, will then coincide with the well known positive Schur property.

Lemma 4.3 *If in a Banach lattice* E *there exists a sequence* $(x_n) \in \ell_p^{weak}(E)$ *with positive terms, which is not norm convergent to 0, then there exists a sequence* $(z_k) \in \ell_p^{weak}(E)$ *such that* $z_n \ge 0$ *for all* n*,* $z_n \wedge z_m = 0$ *for all* $m \ne n$ *and* $\|z_n\| \nrightarrow 0$.

It therefore follows that:

Proposition 4.4 *A Banach lattice* E *has the positive Schur property of order* p *if and only if each disjoint sequence* $(x_n) \in \ell_p^{weak}(E)$ *with positive terms, is norm convergent to 0.*

Definition 4.5 *A Banach space X is said to have the Schur property of order p (briefly, X has the SP_p) if every weakly p-summable sequence is norm convergent to 0.*

It follows from the literature (cf. for instance [4], Proposition 2.1) that every weakly p-summable sequence in a Banach space X is norm convergent to 0 (for $1 \leq p < \infty$) if and only if $\ell_p^{weak}(X) = \ell_p^u(X)$. As is mentioned in Remark 1.1, it is a well-known fact that $\ell_1^{weak}(X) = \ell_1^u(X)$ if and only if X contains no copy of c_0. Thus, we immediately conclude that:

Proposition 4.6 *Let X be a Banach space which contains no copy of c_0. Then X has the SP_1.*

Corollary 4.7 *In each Banach lattice E which contains no copy of c_0, the lattice operations are weakly sequentially* 1*-continuous.*

[0] [4] Aywa, S., Fourie, J.H. "On convergence of sections of sequences in Banach spaces". *Rend.Circ. Mat.Palermo* **II (XLIX)**, (2000)

In general, the weakly sequential continuity of the lattice operations in a Banach lattice is not implied by the weakly sequentially p-continuity of the same, as is illustrated by the following result:

Proposition 4.8 *The space $L_1[0,1]$ has SP_1. Thus, the lattice operations in $L_1[0,1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.*

Proof $L_1[0,1]$ does not contain a copy of c_0; i.e. by Proposition 4.6, it has SP_1. Thus, the lattice operations in $L_1[0,1]$ are weakly sequentially 1-continuous. It is however well-known that the lattice operations in $L_1[0,1]$ are not weakly sequentially continuous. $\square$

Actually, more is known:

Example 4.9

(i) *Let $1 \leq p < \infty$. Recalling that every weak ℓ_1-sequence in an L_p-space is a strong ℓ_r sequence where $r = \max\{p, 2\}$, it follows that any L_p-space has SP_1. Thus, the lattice operations in $L_p[0,1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.*

(ii) *All Banach spaces with finite cotype have SP_1.*

(iii) *A Banach space X has the SP_p (for $1 < p < \infty$) iff each bounded linear operator from $\ell^{p'}$ to X is compact (and X has the SP_1 if each bounded linear operator from c_0 to X is compact). This provides us with an abundance of examples of Banach spaces which have the SP_p for some $1 \leq p < \infty$, but which do not have the Schur property, such as all closed subspaces of ℓ_q (for all $1 < q < p'$) where $1 < p < \infty$. Also, all the ℓ_p-spaces have the SP_1, whereas none of the ℓ_p-spaces (for $p > 1$) have the Schur property.*

On the other hand, we have the following example of a space with the DPP which does not have SP_2:

Example 4.10 *Let (Ω, Σ, μ) be some probability space. The space $L_1(\mu)$ has the DPP (by the Dunford-Pettis Theorem) and thus also has the DPP_p for all $1 \leq p \leq \infty$. By the above discussion, every weakly 2-summable sequence in $L_1(\mu)$ would be a norm null sequence if and only if each bounded linear operator from the sequence space ℓ_2 to $L_1(\mu)$ were compact. This is impossible, since for instance we know that ℓ_2 embeds isometrically in $L_1(\mu)$. Thus, there has to be a weakly 2-summable sequence which is not norm null, showing that $L_1(\mu)$ does not have the SP_2.*

For separable Banach spaces we have:

Theorem 4.11 *A separable Banach space X with the DP^*P_p has the Schur property of order p.*

More examples of L_p-spaces without the Schur property of order p for some choices of p follow from Theorem 6.4.19 in [2]:

(i) L_r does not have SP_p for all $2 \leq p \leq r'$.

(ii) Let $2 < r < \infty$, then L_r does not have SP_p for $p = 2$ or $p = r'$.

[0] [2] Albiac, F., Kalton, N. J.: "Topics in Banach Space Theory".

It is clear that if a Banach lattice E has the SP_p, then the lattice operations are weakly sequentially p-continuous and E has the SP_p^+. On the other hand, if E is a weak p-consistent Banach lattice and E has the SP_p^+, then for each $(x_n) \in \ell_p^{weak}(E)$ we have $(|x_n|) \in \ell_p^{weak}(E)$ and so $\|x_n\| = \||x_n|\| \to 0$ as $n \to \infty$. Thus we have:

Proposition 4.12 *Let E be a weak p-consistent Banach lattice. Then the following are equivalent:*

(i) E has the SP_p.

(ii) E has the SP_p^+.

5 More on p-convergent operators on Banach lattices

Theorem 5.1 *Let E and F be Banach lattices. Suppose the p-convergent operators from E to F satisfy the following domination property: "If S, $T : E \to F$, with $0 \leq S \leq T$ such that T is p-convergent, then likewise S is p-convergent". Then at least one of the following conditions has to hold:*

(a) F has order continuous norm.

(b) The lattice operations in E are weakly sequentially p-continuous.

The following result is a partial converse of Theorem 5.1, assuming a stronger property than in Theorem 5.1(b):

Proposition 5.2 *Let E be a weak p-consistent Banach lattice and F any Banach lattice. If $S, T : E \to F$ are positive operators satisfying $0 \leq S \leq T$ and T is p-convergent, then likewise S is p-convergent.*

When the target space is an AL-space, then we have the following easy characterisation of a p-convergent operator:

Proposition 5.3 *Let E be a Banach lattice and let F be an AL-space. Then the following are equivalent:*

(1) T is p-convergent.

(2) $|Tx_n| \to 0$ as $n \to \infty$ weakly in F for all $(x_i) \in \ell_p^{weak}(E)$.

Proof (1) $\implies$ (2) is clear from $\| |Tx_n| \| = \|Tx_n\| \to 0$ as $n \to \infty$.
To prove (2) $\implies$ (1), observe that the linear functional $e \in F^*$ defined by

$$e(y) := \|y^+\| - \|y^-\|$$

on the AL-space F satisfies $e(|y|) = \|y\|$ for all $y \in F$ (see [3], page 200). Thus,

$$\|Tx_n\| = e(|Tx_n|) \to 0$$

for all $(x_i) \in \ell_p^{weak}(E)$; i.e. T is a p-convergent operator. □

[0] [3] Aliprantis, C. D., Burkinshaw, O. "Positive Operators".

From the previous result we see that:

Proposition 5.4 *Let E be a Banach lattice and let F be an AL-space in which the lattice operations are weakly p-sequentially continuous, then each positive operator $T : E \to F$ is p-convergent.*

Proof Being positive, T is bounded. Thus, if $(x_i) \in \ell_p^{weak}(E)$ is given, then $(Tx_i) \in \ell_p^{weak}(F)$. By assumption, $|Tx_n| \to 0$ weakly. Therefore, by Proposition 5.3, the operator T is p-convergent. $\square$

Remark 5.5 *Let E be a Banach lattice. From Proposition 4.8 and Proposition 5.4 it follows that each positive operator $T : E \to L_1[0,1]$ is 1-convergent.*

Theorem 5.6 *Let E be a Banach lattice. Then, the following assertions are equivalent:*

(1) Each positive operator from E into ℓ_∞ is p-convergent.

(2) E has the SP_p.

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