Trajectorial Martingales, Null Sets, Convergence and Integration

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Comments:

I will present work in progress based on joint work in progress with Alfredo Gonzalez from Mar del Plata National University.

Abstract:

- We extend the well known martingale convergence theorem by replacing the martingale process by a set of trajectories. No apriori given expectation or topology is assumed in such a set (which has no cardinality restrictions). After defining the notion of a property holding a.e. and of a full set one can define a native integral operator.
- Natural and general hypothesis on the trajectory set lead to a convergence theorem incorporating an almost everywhere notion. The results have natural interpretations in terms of portfolios and gambling in a worst case point of view as contrasted to an expectation point of view.

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- We briefly motivate why one may consider weakening the reliance on probability assumptions.
- We emphasize the role of a minmax operator when probabilities are not assumed.
- After pointing out a few simple facts we define a trajectory based martingale (non probabilistic martingale in discrete time).
- We pursue the analogy with standard martingales and prove an a.e. convergence result that also naturally leads to the development of an integration theory.
- Mention some work to be completed.

Stochastic Market Models

$X = \{X_t\}$ is a discrete time stochastic process on $(\Omega, P, \mathcal{F} = \{\mathcal{F}_t\})$, the portfolio value is $V_t^\phi = \phi_t X_t + B_t$ with ϕ_t a predictable process and B_t a riskless bank account that pays 0 interest rates. If no money enters or leaves the portfolio holdings (ϕ_t, B_t) during the life of the portfolio (i.e. self-financing) we have a process transform:

$$V_t^\phi = V_0 + \sum_t \phi_t \Delta_t X \tag{1}$$

where $\Delta_t X \equiv X_{t+1} - X_t$.

Stochastic Arbitrage

$\mathcal{M} = (\Omega, P, \mathcal{F} = \{\mathcal{F}_t\}, X, B)$ (always discrete time) is the market model. There is arbitrage (a riskless profit) in $\mathcal{M}$ if there exists ϕ and T so that

$V_T^{\phi}(w) \geq V_0^{\phi}(w)$ a.e. and
$V_T^{\phi}(w) > V_0^{\phi}(w)$ on $A \in \mathcal{F}_T$ and $P(A) > 0$.

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A main result (FTAP= Fundamental Theorem of Asset Pricing) says that $\mathcal{M}$ is arbitrage free iff X (discounted) is a martingale relative to $Q \sim P$.

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- So, we ask: what is the structure of the trajectory set of a martingale process that can guarantee markets without riskless profits? You will see that one can develop a theory (work in progress) based on trajectories where probabilities are replaced for uncertainty (just meaning that one of many alternatives may occur).
- The basic property is that of a martingale difference $\mathbb{E}(\Delta_t X|\mathcal{F}_t) = 0$ which requires that:
  if $Q(\Delta_t X > 0|\mathcal{F}_t) > 0$ then $Q(\Delta_t X) < 0|\mathcal{F}_t) > 0$ as well. Or if $Q(\Delta_t X > 0|\mathcal{F}_t) = 0$ then $Q(\Delta_t X < 0|\mathcal{F}_t) = 0$ as well.

Trajectory Sets (finite time portfolios)

- A trajectory set $\mathcal{S}$ is a set of sequences $S = \{S_t\}_{t \geq 0}$. A portfolio set $\mathcal{H}$ is a set of sequences $H = \{H_t\}$, $H_t : \mathcal{S} \to \mathbb{R}$ $H_t(S) = H_t(S_0, \ldots, S_t)$. For the time being we assume $\exists$ $N_H = N_H(S)$ and $H_i(S) = H_{N_H}(S) \;\; \forall\, i \geq N_H$ (one can take N_H constant for simplicity).

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- For $Z : \mathcal{S} \to \mathbb{R}$ define

$$\overline{V}(Z) \equiv \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H - 1} H_i(S) \; \Delta_i S]. \tag{2}$$

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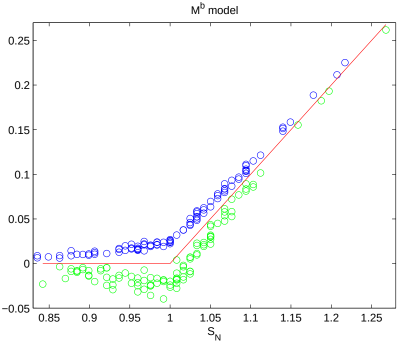
- A trajectory market $\mathcal{M} \equiv \mathcal{S} \times \mathcal{H}$ is (globally) 0-neutral if

$$\overline{V}(0) = 0. \tag{3}$$



M^b model
0.25
0.2
0.15
0.1
0.05
0
−0.05
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
S_N

Trajectories from Martingales and Minmax

- **Trajectories from Martingales**: let $\mathcal{S} = \{\{S_t\} : \exists\ w \quad S_t = X_t(w)\}$, $\mathcal{H} = \{\{H_t\} : H_t(S) = \phi_t(w), \phi \in \mathcal{F}_t\}$

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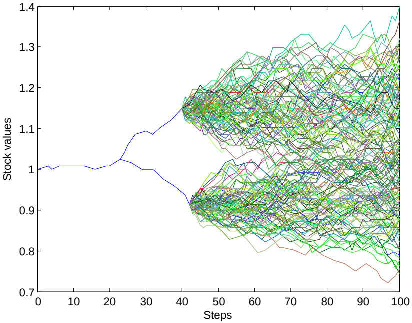
- Consider N_H constant (or, more generally, a stopping time), because $[-\sum_{i=0}^{N_H-1} H_i(S)\ \Delta_i S] \leq \sup_{S\in\mathcal{S}}[-\sum_{i=0}^{N_H-1} H_i(S)\ \Delta_i S]$ and $0 = \mathbb{E}([-\sum_{i=0}^{N_H-1} H_i(S)\ \Delta_i S]$ and $0 \in \mathcal{H}$.

$$\overline{V}(0) \equiv \inf_{H\in\mathcal{H}} \sup_{S\in\mathcal{S}}[-\sum_{i=0}^{N_H-1} H_i(S)\ \Delta_i S] = 0. \tag{4}$$

Conditional Sets of Trajectories

For $S \in \mathcal{S}$ and $j \geq 0$ define

$$\mathcal{S}_{(S,k)} = \{\hat{S} \in \mathcal{S} : \hat{S}_j = S_j,\ 0 \leq j \leq k\}. \tag{5}$$



1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
Stock values
0
10
20
30
40
50
60
70
80
90
100
Steps

Conditional Up Down Property

$\mathcal{S}$ is said to satisfy the conditional up down property if for S and j fixed

$$\sup_{\hat{S} \in \mathcal{S}_{(S,j)}} (\hat{S}_{j+1} - S_j) > 0, \quad \text{and} \quad \inf_{\hat{S} \in \mathcal{S}_{(S,j)}} (\hat{S}_{j+1} - S_j) < 0, \tag{6}$$

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Define also the (local at (S,t)) 0-neutral property

$$\sup_{S' \in \mathcal{S}_{(S,t)}} (S'_{t+1} - S_t) \geq 0 \;\; \text{and} \;\; \inf_{S' \in \mathcal{S}_{(S,t)}} (S'_{t+1} - S_t) \leq 0 \tag{8}$$

0-Neutrality a Generalization of the Martingale Property

- A trajectory set $\mathcal{S}$ satisfying the local up-down property at every (S, t) is our notion of trajectory based martingale. A more general trajectory set $\mathcal{S}$ is when the local 0-neutral property is satisfied.

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- A locally 0-neutral trajectory set defines naturally a notion of integration with an associated a.e. notion, we will argue that $\lim_{n\to\infty} S_n$ converges a.e. More generally, it follows that $\lim_{n\to\infty} \sum_{i=0}^{n} H_i(S)(S_{i+1} - S_i)$ converges a.e.

Key Property

- The following operator generalizes $\overline{V}$ to account for unbounded time.

Definition

$$\overline{W}(Z) = \overline{W}(Z, \mathcal{M}) = \inf_{H \in \mathcal{H}} \left[\sup_{S \in \mathcal{S}} \left(Z(S) - \liminf_{n \to \infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S \right) \right]. \quad (9)$$

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- If $\overline{W}(0) = 0$ then $||\mathbf{1}_{\mathcal{S}}|| = 1$ (this will imply that the set of convergence of a trajectorial martingale is a full set.)
- How can we guarantee $\overline{W}(0) = 0$? We answer this question at the end of the talk.

Integration Theory Associated to $\mathcal{S}$ and $\overline{W}$

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- Daniell's approach to integration (that, we recall, yields the usual Lebesgue's integration) requires: $\mathcal{E}$, a vector lattice of functions and I a linear and order preserving functional on I that satisfies $h_n(x) \searrow 0$ then $I(h_n) \searrow 0$ for $h_n \in \mathcal{E}$ (continuity at 0). Daniell's construction then extends by continuity from $\mathcal{E}$ to a space L^1 and the usual, Lebesgue, properties of the integral are then obtained.

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- We will take $I = \overline{W}|_{\mathcal{E}}$ and $\mathcal{E}$ the set of "attainable" functions:

$$\mathcal{E} \equiv \{Z : \mathcal{S} \to \mathbb{R} : \exists H \; Z(S) = V^Z + \sum_{i=0}^{n-1} H_i(S)\Delta_i S \;\; \text{for constant } n\}, \tag{10}$$

here $H = \{H_i\}$.

Integration Theory Associated to $\mathcal{S}$ and $\overline{W}$

- $\mathcal{E}$,as above, is a vector space over $\mathbb{R}$ and if $\overline{W}(0) = 0$ we have $\underline{W}(Z) = \overline{W}(Z) = V^Z$ and so $\overline{W}$ is linear and monotone on $\mathcal{E}$ but this space is not a lattice.

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- In such setting, Leinert (1980) defines an outer functional $\bar{I}$, a functional version of Caratheodory outer measure and obtains a weaker theory of integration that generalizes Daniell's construction. Some limit theorems being valid (Beppo-Levi and monotone convergence) but others do not hold with the same generality. The integral is not classical, i.e. it is not necessarily associated to a σ-algebra but it becomes so if some (weak) lattice type properties are in effect.

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- We have to modify Leinert's approach somehow, the 0-neutral hypothesis on $\mathcal{S}$ (here taking the form $\overline{W}(0) = 0$) is used to establish a needed continuity property of I.

Integration Theory Associated to $\mathcal{S}$ and $\overline{W}$

- In essence $\overline{W}(\cdot)$ is extended by continuity from its natural domain where it is linear to limits of such attainable functions (i.e. elements of $\mathcal{E}$) the final result is: such an outer functional is a super-replication functional that upperbounds upcrossings. This shows that the set of infinite upcrosses is a null set and so proving the convergence result.

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- Note: the use of super-replication ideas to construct a measure appears in work of Vovk.

Integration Theory Associated to $\mathcal{S}$ and $\overline{W}$

- Set $\mathcal{P} = \{f : \mathcal{S} \to [0, \infty]\}$
- Result: $|| \cdot || = \overline{W}(\cdot)$ is countable subadditive on $\mathcal{P}$.

Definition

For $g : \mathcal{S} \to [-\infty, \infty]$: A function g is a *null function* if $\|g\| = 0$, a subset $E \subset \mathcal{S}$ is a *null set* if $\|\mathbf{1}_E\| = 0$. A set E is full if $\|\mathbf{1}_E\| = 1$. A property holds a.e. if it holds in the complement of a null set (actually this is to keep with tradition but here one also needs to check that such complement is full).

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- 1) $||g|| = 0 \iff g = 0$ *a.e.* 2) Countable unions of null sets are null sets 3) $f = g$ *a.e.* $\implies ||f|| = ||g||$.

Integration Theory Associated to $\mathcal{S}$ and $\overline{W}$

- One can show that $I = \overline{W}|_{\mathcal{E}}$ is bounded on $\mathcal{E}$. One obtains an integral by continuity extension.
- Let $\mathcal{E}' \equiv \{f \in \mathcal{E} : ||f|| < \infty\}$ and let $\mathcal{L}^1$ be its norm closure (it ends up being a complete space).

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- Let $\int f$, $f \in \mathcal{L}^1$, be the continuous extension, from $\mathcal{E}'$ to $\mathcal{L}^1$, of I.
- I will not show details of the construction but mention that there are two such related integrals that differ on hypothesis on $\mathcal{H}$ and one of them satisfying only a weak form of the monotone convergence theorem. Both lead to complete $\mathcal{L}^1$ spaces.

Theorem:

$$\lim_{n\to\infty} S_n \;\text{ converges a.e. on }\; \mathcal{S}, \tag{11}$$

For simplicity take $S_i \geq 0$, define $A_n^k \equiv \{S \in \mathcal{S}: \; U_n(S) \geq k\}$ (U_n are upcrossings over $[a, b]$), $A^k \equiv \cup_{n\geq 1} A_n^k$ and $A \equiv \cap_{k\geq 1} A^k$. By the upcrossing inequality:

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$$\mathbf{1}_A(S) \leq \; \mathbf{1}_{A^k}(S) \leq \frac{a}{k(b-a)} + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{k(b-a)} D_i(S) \; \Delta_i S, \quad \forall S \in \mathcal{S}. \tag{12}$$

Since $\frac{a}{k(b-a)} + \sum_{i=0}^{n-1} \frac{1}{k(b-a)} D_i(S) \Delta_i S \geq 0, \quad \forall S \in \mathcal{S}$, by definition of $|| \cdot ||$, we have

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It then follows that $||1_{\cup_i\{U_{\infty,a_i,b_i}=\infty\}}|| = 0$ where $[a_i, b_i]$ is an arbitrary countable collection of intervals.

From $||1_{\cup_i\{U_{\infty,a_i,b}=\infty\}}|| = 0$ it follows that $S_\infty \equiv \lim_{n\to\infty} S_n$ exists in $\overline{\mathbb{R}}$ a.e. in $S \in \mathcal{S}$ (this holds by the usual/general arguments employed in the martingale convergence theorem).

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$$A_\infty \equiv \{S \in \mathcal{S} : \; S_\infty = \infty\} \subseteq \tag{13}$$

$$\{S \in \mathcal{S} : \; \exists \; M = M(S), \quad S_n \geq \frac{1}{\epsilon}, \quad \text{if} \quad n \geq M\} \equiv A_\epsilon. \tag{14}$$

If $S \in A_\epsilon$, then $s_0 + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \Delta_i S \geq \frac{1}{\epsilon}$, consequently for all $S \in \mathcal{S}$,

$$\mathbf{1}_{A_\infty} \leq \mathbf{1}_{A_\epsilon} \leq \epsilon s_0 + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \epsilon \Delta_i S. \tag{15}$$

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$$\mathbf{1}_{A_\infty} \leq \mathbf{1}_{A_\epsilon} \leq \epsilon s_0 + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \epsilon \Delta_i S. \tag{15}$$

Since $\epsilon s_0 + \sum_{i=0}^{n-1} \epsilon \Delta_i S = \epsilon S_n \geq 0$ it follows by definition of $|| \cdot ||$ that

$$||\mathbf{1}_{A_\infty}|| \leq ||\mathbf{1}_{A_\epsilon}|| \leq \epsilon s_0.$$

So $||\mathbf{1}_{A_\infty}|| = 0$.

Key Property

- Recall

$$\overline{W}(Z) = \overline{W}(Z, \mathcal{M}) = \inf_{H \in \mathcal{H}} \left[\sup_{S \in \mathcal{S}} \left(Z(S) - \liminf_{n \to \infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S \right) \right]. \tag{16}$$

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- How can we guarantee $\overline{W}(0) = 0$? The notion needed is:

Contrarian Trajectories

Definition

Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ a market, $F = (F_i)_{i \geq 0}$ a sequence of functions (not necessarily in $\mathcal{H}$) and $\epsilon > 0$ given. $S^{\epsilon} \in \mathcal{S}$ is called an *ϵ-contrarian trajectory* (CT) for F, if for any $n \geq 1$

$$\sum_{i=0}^{n-1} F_i(S^{\epsilon}) \ \ \Delta_i S^{\epsilon} < \sum_{i=0}^{n-1} \frac{\epsilon}{2^{i+1}} < \epsilon. \tag{17}$$

We will say that $\mathcal{M}$ *admits contrarian trajectories*, if for any $H \in \mathcal{H}$ and $\epsilon > 0$, there exist an ϵ-contrarian trajectory for H.

Contrarian Trajectories

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- Contrarian trajectories in the limit can be added to the trajectory set but it is hard to control de size of the set of the added trajectories. Stochastic processes seem to have them but they are not prominent and hide behind continuity properties of a measure.
- It will take me some quality time to describe general and natural conditions on trajectory sets providing existence of CT but it can be done.

Extension to Trajectorial Transform and Stochastic Integration

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- Similarly to the construction of the integral, there is the possibility that the analogue of stochastic integration can also be obtained.
- Thank you!