$u\tau$-Topologies in locally solid Riesz spaces

Eduard Emel'yanov

Middle East Technical University, Ankara, Turkey

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Content

Introduction

- All Riesz spaces in the talk are supposed to be real and Archimeadean.
- Let "$\xrightarrow{\mathrm{c}}$" be a convergence on a Riesz space X which agrees with linear and lattice operations in X in the following sence:

$$x_\alpha \equiv x \Rightarrow x_\alpha \xrightarrow{\mathrm{c}} x; \tag{1}$$

and

$$X \ni x_\alpha \xrightarrow{\mathrm{c}} x, \;\; X \ni y_\alpha \xrightarrow{\mathrm{c}} y, \;\; \mathbb{R} \ni r_\alpha \to r$$

imply

$$r_\alpha \cdot x_\alpha + y_\alpha \xrightarrow{\mathrm{c}} r \cdot x + y, \tag{2}$$

$$r_\alpha \cdot x_\alpha \wedge y_\alpha \xrightarrow{\mathrm{c}} r \cdot x \wedge y. \tag{3}$$

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- As basic examples of such a convergence, we keep in mind o-convergence, ru-convergence, τ-convergence (if X is a locally solid Riesz space with its locally solid topology τ), and m-convergence (if $X = (X, m)$ is a multi-normed Riesz space).
- We say that $x_\alpha \xrightarrow{\text{uc}} x$ in X if $|x_\alpha - x| \wedge u \xrightarrow{\text{c}} 0$ for any $u \in X_+$, where "$\xrightarrow{\text{uc}}$" stands for unbounded c-convergence.
- It is immediate to see that uc-convergence is also agreed with linear and lattice operations in X, and uuc-convergence coincides with uc-convergence.

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- Notice that, in $L_1[0,1]$, the uo-convergence of sequences is equivalent to the almost everywhere convergence.
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- Notice also that *ru*-convergence is topological iff X has a strong order unit [7, Thm.5].
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- Another natural question consists in finding conditions under which *uc*-convergence agrees with *c*-convergence.
- The main topic of this talk is related to the case when our *c*-convergence is already topological.
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Several results on $u\tau$-topology

We begin with two equivalent definitions of $u\tau$-topology (cf. [7, 19, 9]). In the case, when c-convergence is topological, say c-convergence coincides with τ-convergence produced by a locally solid topology τ on X, the uc-convergence is topological too, and a zero-base of the corresponding locally solid topology $u\tau$ can be taken as follows [7, 19]:

$$U_{w,V} = \{x \in X : |x| \wedge w \in V\} \quad (w \in X_+, V \in \tau(0)). \tag{4}$$

Notice also that, as the topology τ is generated by a family $R = \{\pi_i\}_{i\in I}$ of Riesz pseudo-semi-norms, the corresponding $u\tau$-topology can be equivalently defined [9] as the topology generated by the family $u(R) = \{\pi_{i,w}\}_{i\in I, w\in X_+}$ of the following Riesz pseudo-semi-norms:

$$\pi_{i,w}(x) := \pi_i(|x| \wedge w) \quad (i \in I, w \in X_+, x \in X). \tag{5}$$

The following several results have been proven in [7].

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The next theorem [7, Thm.1] should be compared with Lemma 2.1. in [16], where it was proved for Banach lattices.

Theorem 1

Let (X, τ) be a locally solid Riesz space, where τ is generated by a family $(\pi_j)_{j\in J}$ of Riesz pseudo-semi-norms. Let $j \in J$, $w \in X_+$, and $\varepsilon > 0$. Define a zero-neighboorhood $V_{j,w,\varepsilon} \in u\tau(0)$ as follows:

$$V_{j,w,\varepsilon} := \{x \in X : \pi_j(|x| \wedge w) < \varepsilon\}.$$

Then, either $V_{j,w,\varepsilon}$ is contained in $[-w, w]$, or $V_{j,w,\varepsilon}$ contains a non-trivial ideal.

For the next result, we need to remind the following definition wich is due to Schaefer.

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Definition 2

An element $0 \neq e \in X_+$ of a topological Riesz space (X, τ) is called a quasi-interior point, if the principal ideal I_e is τ-dense in X.

The next theorem [7, Thm. 2] extends Theorem 3.1 in [16], where it was proved for Banach lattices.

Theorem 3

Let (X, τ) be a sequentially complete locally solid Riesz space. Let $e \in X_+$. The following are equivalent:

1. *e is a quasi-interior point;*
2. *for every net x_α in X_+, if $x_\alpha \wedge e \xrightarrow{\tau} 0$ then $x_\alpha \xrightarrow{u\tau} 0$;*
3. *for every sequence x_n in X_+, if $x_n \wedge e \xrightarrow{\tau} 0$ then $x_n \xrightarrow{u\tau} 0$.*

Now, we need the following generalization of quasi-interior points.

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A pairwise disjoint system $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ of non-zero positive elements of a topological Riesz space (X, τ) is said to be a topological orthogonal system, if the ideal I_Q generated by Q is τ-dense in X.

The next theorem [7, Thm. 7] extends the above theorem to the case when our Riesz space X is possessing just a topological orthogonal system.

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Let (X, τ) be a locally solid Riesz space, and $Q = \{e_\gamma\}_{\gamma\in\Gamma}$ be a topological orthogonal system of (X, τ). Then $x_\alpha \xrightarrow{u\tau} 0$ iff $|x_\alpha| \wedge e_\gamma \xrightarrow{\tau} 0$ for every $\gamma \in \Gamma$.

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Let (X, τ) be a complete metrizable locally solid Riesz space. If X has a countable topological orthogonal system, then the $u\tau$-topology is metrizable.

- Notice that it is still unknown whether or not the converse of this theorem is true.

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When *uc*-convergence coincides with *c*-convergence

- Clearly, c-convergence always implies uc-convergence, and they coincide, when the Riesz space X under the consideration is finite dimensional.
- Here we discuss for several types of c-convergence the question: whether or not c-convergence coincides with uc-convergence only if $\dim(X) < \infty$? To avoid trivial cases, suppose throughout this section that $\dim(X) = \infty$.
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- It is known that *o*-convergence in X is not topological [6, Thm. 1]. However, *uo*-convergence in X is topological, when X is discrete [6, Thm. 2].
- We do not know any examples of non-discrete X in which *uo*-convergence in X is topological.
- Since *uo*-convergence of a net $x_\alpha \in X$ coincides with *uo*-convergence of x_α in X^δ by [13, Thm. 3.2], where X^δ is the Dedekind completion of X, we may also assume that X is Dedekind complete.

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- Accordingly to the remark above, we exclude the case when $X \equiv c_{00}(\Omega)$. Indeed, *uo*-convergence in $c_{00}(\Omega)$ is topological, but *o*-convergence is not.
- Therefore, we may suppose that, for some $u \in X_+$, there is a sequence of pair-wise disjoint vectors $X_+ \ni u_n \neq 0$ such that $u = \sup_n u_n$.
- Now, we construct the following net $(x_\Delta)_{\Delta \in \mathcal{P}_{fin}(\mathbb{N})}$:

$$x_\Delta := |\Delta| \cdot \sup_{n \notin \Delta} u_n \quad (\Delta \in \mathcal{P}_{fin}(\mathbb{N})), \tag{6}$$

where $\mathcal{P}_{fin}(\mathbb{N})$ is ordered by inclusion.

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$$x_\Delta := |\Delta| \cdot \sup_{n \notin \Delta} u_n \quad (\Delta \in \mathcal{P}_{fin}(\mathbb{N})), \tag{6}$$

where $\mathcal{P}_{fin}(\mathbb{N})$ is ordered by inclusion.

When *uc*-convergence coincides with *c*-convergence

- Accordingly to the remark above, we exclude the case when $X \equiv c_{00}(\Omega)$. Indeed, *uo*-convergence in $c_{00}(\Omega)$ is topological, but *o*-convergence is not.
- Therefore, we may suppose that, for some $u \in X_+$, there is a sequence of pair-wise disjoint vectors $X_+ \ni u_n \neq 0$ such that $u = \sup_n u_n$.
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When *uc*-convergence coincides with *c*-convergence

- It can be easiely seen that the net x_Δ is not eventually o-bounded in X. Thus, x_Δ is not o-convergent in X.
- From the other hand, $x_\Delta \xrightarrow{\text{uo}} 0$ in I_u and hence $x_\Delta \xrightarrow{\text{uo}} 0$ in X.
- Thus, we get the following result which is included in [9].

Proposition 7

Let X be an Archimedean Riesz space. Then uo-convergence agrees with o-convergence in X iff $\dim(X) < \infty$.

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When *uc*-convergence coincides with *c*-convergence

- The case of m-convergence and, therefore of τ-convergence, is surprisingly different.
- Namely, for $X = c_{00}$ equipped with the multi-norm $\mathcal{M} = \{m_n\}_{n\in\mathbb{N}}$:

$$m_k(x) := x_k \qquad (x = (x_n)_n \in c_{00}),$$

we have that $x_\alpha \xrightarrow{\text{um}} 0$ iff $x_\alpha \xrightarrow{\text{m}} 0$ iff $x_\alpha \to 0$ coordinatewise.

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