Triangularizability of trace-class operators with increasing spectrum

Roman Drnovšek

University of Ljubljana, Slovenia

Positivity IX, Edmonton, July 2017

Let μ be a positive measure on a set X such that $L^2(X,\mu)$ is a separable complex Hilbert space.

For each $\phi \in L^\infty(X,\mu)$, we define the multiplication operator M_ϕ on $L^2(X,\mu)$ by $M_\phi(f) = \phi f$.

An operator P on $L^2(X,\mu)$ is called a *standard projection corresponding to a measurable set* $E \subseteq X$ if it is the multiplication operator by the characteristic function χ_E of E.

In this case its range $\operatorname{ran} P$ can be identified with the Hilbert space $L^2(E,\mu|_E)$, and it is said to be a *standard subspace* or a *closed ideal* of $L^2(X,\mu)$.

If such a subspace is non-trivial and invariant under an operator T on $L^2(X,\mu)$, we say that T is *decomposable* or *ideal-reducible*.

Let μ be a positive measure on a set X such that $L^2(X,\mu)$ is a separable complex Hilbert space.
For each $\phi \in L^\infty(X,\mu)$, we define the multiplication operator M_ϕ on $L^2(X,\mu)$ by $M_\phi(f) = \phi f$.
An operator P on $L^2(X,\mu)$ is called a *standard projection corresponding to a measurable set* $E \subseteq X$ if it is the multiplication operator by the characteristic function χ_E of E.
In this case its range $\operatorname{ran} P$ can be identified with the Hilbert space $L^2(E,\mu|_E)$, and it is said to be a *standard subspace* or a *closed ideal* of $L^2(X,\mu)$.
If such a subspace is non-trivial and invariant under an operator T on $L^2(X,\mu)$, we say that T is *decomposable* or *ideal-reducible*.

Let μ be a positive measure on a set X such that $L^2(X,\mu)$ is a separable complex Hilbert space.
For each $\phi \in L^\infty(X,\mu)$, we define the multiplication operator M_ϕ on $L^2(X,\mu)$ by $M_\phi(f) = \phi f$.
An operator P on $L^2(X,\mu)$ is called a *standard projection corresponding to a measurable set* $E \subseteq X$ if it is the multiplication operator by the characteristic function χ_E of E.
In this case its range $\operatorname{ran} P$ can be identified with the Hilbert space $L^2(E,\mu|_E)$, and it is said to be a *standard subspace* or a *closed ideal* of $L^2(X,\mu)$.
If such a subspace is non-trivial and invariant under an operator T on $L^2(X,\mu)$, we say that T is *decomposable* or *ideal-reducible*.

Let μ be a positive measure on a set X such that $L^2(X,\mu)$ is a separable complex Hilbert space.

For each $\phi \in L^\infty(X,\mu)$, we define the multiplication operator M_ϕ on $L^2(X,\mu)$ by $M_\phi(f) = \phi f$.

An operator P on $L^2(X,\mu)$ is called a *standard projection corresponding to a measurable set* $E \subseteq X$ if it is the multiplication operator by the characteristic function χ_E of E.

In this case its range $\mathrm{ran}\, P$ can be identified with the Hilbert space $L^2(E,\mu|_E)$, and it is said to be a *standard subspace* or a *closed ideal* of $L^2(X,\mu)$.

If such a subspace is non-trivial and invariant under an operator T on $L^2(X,\mu)$, we say that T is *decomposable* or *ideal-reducible*.

Let μ be a positive measure on a set X such that $L^2(X,\mu)$ is a separable complex Hilbert space.
For each $\phi \in L^\infty(X,\mu)$, we define the multiplication operator M_ϕ on $L^2(X,\mu)$ by $M_\phi(f) = \phi f$.
An operator P on $L^2(X,\mu)$ is called a *standard projection corresponding to a measurable set* $E \subseteq X$ if it is the multiplication operator by the characteristic function χ_E of E.
In this case its range $\operatorname{ran} P$ can be identified with the Hilbert space $L^2(E,\mu|_E)$, and it is said to be a *standard subspace* or a *closed ideal* of $L^2(X,\mu)$.
If such a subspace is non-trivial and invariant under an operator T on $L^2(X,\mu)$, we say that T is *decomposable* or *ideal-reducible*.

An operator T on $L^2(X,\mu)$ *admits a standard triangularization* or T is *completely decomposable* or *ideal-triangularizable* if we can find a totally ordered set Λ and an increasing family $\{P_\lambda\}_{\lambda\in\Lambda}$ of standard projections such that $\{\operatorname{ran} P_\lambda\}_{\lambda\in\Lambda}$ is a maximal increasing family of standard subspaces that are all invariant under T.

An operator T on $L^2(X,\mu)$ *has increasing spectrum relative to standard compressions* if

$$\sigma(PT|_{\operatorname{ran} P}) \subseteq \sigma(QT|_{\operatorname{ran} Q})$$

whenever P and Q are standard projections with $\operatorname{ran} P \subseteq \operatorname{ran} Q$. When this condition is required only for finite-dimensional standard projections P and Q, the operator T is said to *have increasing spectrum relative to finite-dimensional standard compressions*.

An operator T on $L^2(X,\mu)$ *admits a standard triangularization* or T is *completely decomposable* or *ideal-triangularizable* if we can find a totally ordered set Λ and an increasing family $\{P_\lambda\}_{\lambda\in\Lambda}$ of standard projections such that $\{\operatorname{ran} P_\lambda\}_{\lambda\in\Lambda}$ is a maximal increasing family of standard subspaces that are all invariant under T.

An operator T on $L^2(X,\mu)$ *has increasing spectrum relative to standard compressions* if

$$\sigma(PT|_{\operatorname{ran} P}) \subseteq \sigma(QT|_{\operatorname{ran} Q})$$

whenever P and Q are standard projections with $\operatorname{ran} P \subseteq \operatorname{ran} Q$.
When this condition is required only for finite-dimensional standard projections P and Q, the operator T is said to *have increasing spectrum relative to finite-dimensional standard compressions*.

An operator T on $L^2(X,\mu)$ *admits a standard triangularization* or T is *completely decomposable* or *ideal-triangularizable* if we can find a totally ordered set Λ and an increasing family $\{P_\lambda\}_{\lambda\in\Lambda}$ of standard projections such that $\{\operatorname{ran} P_\lambda\}_{\lambda\in\Lambda}$ is a maximal increasing family of standard subspaces that are all invariant under T.

An operator T on $L^2(X,\mu)$ *has increasing spectrum relative to standard compressions* if

$$\sigma(PT|_{\operatorname{ran} P}) \subseteq \sigma(QT|_{\operatorname{ran} Q})$$

whenever P and Q are standard projections with $\operatorname{ran} P \subseteq \operatorname{ran} Q$. When this condition is required only for finite-dimensional standard projections P and Q, the operator T is said to *have increasing spectrum relative to finite-dimensional standard compressions*.

Theorem (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Let μ be the counting measure on a set X. If an operator T on $L^2(X,\mu)$ has increasing spectrum relative to finite-dimensional standard compressions, then it is ideal-triangularizable.

Theorem (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Let T be an operator on $L^2(X,\mu)$ of rank $n \in \mathbb{N}$. If T has increasing spectrum relative to standard compressions, then it admits a standard triangularization. Furthermore, there is a chain of projections

$$0 = P_0 < P_1 < \cdots < P_{3n-1} < P_{3n} = I,$$

whose ranges are all invariant under T, such that

$$(P_j - P_{j-1})T(P_j - P_{j-1}) = 0$$

whenever $P_j - P_{j-1}$ has rank more than one.

Theorem (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Let μ be the counting measure on a set X. If an operator T on $L^2(X,\mu)$ has increasing spectrum relative to finite-dimensional standard compressions, then it is ideal-triangularizable.

Theorem (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Let T be an operator on $L^2(X,\mu)$ of rank $n \in \mathbb{N}$. If T has increasing spectrum relative to standard compressions, then it admits a standard triangularization. Furthermore, there is a chain of projections

$$0 = P_0 < P_1 < \cdots < P_{3n-1} < P_{3n} = I,$$

whose ranges are all invariant under T, such that

$$(P_j - P_{j-1})T(P_j - P_{j-1}) = 0$$

whenever $P_j - P_{j-1}$ has rank more than one.

Question (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Suppose that K is a compact operator on $L^2(X, \mu)$ that has increasing spectrum relative to standard compressions. Does K admit a standard triangularization? In particular, is K ideal-reducible?

Theorem (de Pagter, 1986)

A quasinilpotent compact positive operator K on a Banach lattice of dimension at least two has a nontrivial invariant closed ideal.

An affirmative answer to Question would extend de Pagter's theorem in the case of the Banach lattice $L^2(X, \mu)$. Namely, it is easy to see that positivity of K implies that the operator PKP is quasinilpotent for each standard projection P, so that K has increasing spectrum in this case.

Question (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Suppose that K is a compact operator on $L^2(X,\mu)$ that has increasing spectrum relative to standard compressions. Does K admit a standard triangularization? In particular, is K ideal-reducible?

Theorem (de Pagter, 1986)

A quasinilpotent compact positive operator K on a Banach lattice of dimension at least two has a nontrivial invariant closed ideal.

An affirmative answer to Question would extend de Pagter's theorem in the case of the Banach lattice $L^2(X,\mu)$. Namely, it is easy to see that positivity of K implies that the operator PKP is quasinilpotent for each standard projection P, so that K has increasing spectrum in this case.

Question (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Suppose that K is a compact operator on $L^2(X,\mu)$ that has increasing spectrum relative to standard compressions. Does K admit a standard triangularization? In particular, is K ideal-reducible?

Theorem (de Pagter, 1986)

A quasinilpotent compact positive operator K on a Banach lattice of dimension at least two has a nontrivial invariant closed ideal.

An affirmative answer to Question would extend de Pagter's theorem in the case of the Banach lattice $L^2(X,\mu)$. Namely, it is easy to see that positivity of K implies that the operator PKP is quasinilpotent for each standard projection P, so that K has increasing spectrum in this case.

Question (Marcoux, Mastnak, Radjavi, J. Funct. Anal. 2009)

Suppose that K is a compact operator on $L^2(X,\mu)$ that has increasing spectrum relative to standard compressions. Does K admit a standard triangularization? In particular, is K ideal-reducible?

Theorem (de Pagter, 1986)

A quasinilpotent compact positive operator K on a Banach lattice of dimension at least two has a nontrivial invariant closed ideal.

An affirmative answer to Question would extend de Pagter's theorem in the case of the Banach lattice $L^2(X,\mu)$. Namely, it is easy to see that positivity of K implies that the operator PKP is quasinilpotent for each standard projection P, so that K has increasing spectrum in this case.

We consider Question for trace-class kernel operators.

An operator K on $L^2(X,\mu)$ is called a *kernel operator* if there exists a measurable function $k : X \times X \to \mathbb{C}$ such that, for every $f \in L^2(X,\mu)$, the equality

$$(Kf)(x) = \int_X k(x,y)f(y)d\mu(y)$$

holds for almost every $x \in X$. The function k is the *kernel* of the operator K.

The kernel operator K is positive if and only if its kernel k is nonnegative almost everywhere.

If the kernel operator K with kernel k has the modulus $|K|$, then the kernel of $|K|$ is equal to $|k|$ almost everywhere.

We consider Question for trace-class kernel operators.
An operator K on $L^2(X,\mu)$ is called a *kernel operator* if there exists a measurable function $k : X \times X \to \mathbb{C}$ such that, for every $f \in L^2(X,\mu)$, the equality

$$(Kf)(x) = \int_X k(x,y)f(y)d\mu(y)$$

holds for almost every $x \in X$. The function k is the *kernel* of the operator K.
The kernel operator K is positive if and only if its kernel k is nonnegative almost everywhere.
If the kernel operator K with kernel k has the modulus $|K|$, then the kernel of $|K|$ is equal to $|k|$ almost everywhere.

We consider Question for trace-class kernel operators.
An operator K on $L^2(X,\mu)$ is called a *kernel operator* if there exists a measurable function $k : X \times X \to \mathbb{C}$ such that, for every $f \in L^2(X,\mu)$, the equality

$$(Kf)(x) = \int_X k(x,y)f(y)d\mu(y)$$

holds for almost every $x \in X$. The function k is the *kernel* of the operator K.
The kernel operator K is positive if and only if its kernel k is nonnegative almost everywhere.
If the kernel operator K with kernel k has the modulus $|K|$, then the kernel of $|K|$ is equal to $|k|$ almost everywhere.

We consider Question for trace-class kernel operators.
An operator K on $L^2(X,\mu)$ is called a *kernel operator* if there exists a measurable function $k : X \times X \to \mathbb{C}$ such that, for every $f \in L^2(X,\mu)$, the equality

$$(Kf)(x) = \int_X k(x,y)f(y)d\mu(y)$$

holds for almost every $x \in X$. The function k is the *kernel* of the operator K.
The kernel operator K is positive if and only if its kernel k is nonnegative almost everywhere.
If the kernel operator K with kernel k has the modulus $|K|$, then the kernel of $|K|$ is equal to $|k|$ almost everywhere.

Given a compact operator T on $L^2(X,\mu)$, let $\{s_j(T)\}_j$ be a decreasing sequence of singular values of T, i.e., the square roots of the eigenvalues of the self-adjoint operator T^*T, where where T^* denotes the adjoint of T.

If $\sum_j s_j(T) < \infty$, the operator T is said to be a *trace-class* operator. In this case, the *trace of T* is defined by

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Tf_n, f_n \rangle,$$

where $\{f_n\}_{n=1}^{\infty}$ is any orthonormal basis of $L^2(X,\mu)$.

By Lidskii's Theorem, the trace of a trace-class operator T is equal to the sum of all eigenvalues of T counting algebraic multiplicity.

Given a compact operator T on $L^2(X, \mu)$, let $\{s_j(T)\}_j$ be a decreasing sequence of singular values of T, i.e., the square roots of the eigenvalues of the self-adjoint operator T^*T, where where T^* denotes the adjoint of T.
If $\sum_j s_j(T) < \infty$, the operator T is said to be a *trace-class* operator. In this case, the *trace of T* is defined by

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Tf_n, f_n \rangle,$$

where $\{f_n\}_{n=1}^{\infty}$ is any orthonormal basis of $L^2(X, \mu)$.
By Lidskii's Theorem, the trace of a trace-class operator T is equal to the sum of all eigenvalues of T counting algebraic multiplicity.

Given a compact operator T on $L^2(X,\mu)$, let $\{s_j(T)\}_j$ be a decreasing sequence of singular values of T, i.e., the square roots of the eigenvalues of the self-adjoint operator T^*T, where where T^* denotes the adjoint of T.
If $\sum_j s_j(T) < \infty$, the operator T is said to be a *trace-class* operator. In this case, the *trace of T* is defined by

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Tf_n, f_n \rangle,$$

where $\{f_n\}_{n=1}^{\infty}$ is any orthonormal basis of $L^2(X,\mu)$.
By Lidskii's Theorem, the trace of a trace-class operator T is equal to the sum of all eigenvalues of T counting algebraic multiplicity.

Let K be a trace-class kernel operator on $L^2[0,1]$ with a continuous kernel $k : [0,1] \times [0,1] \to \mathbb{C}$. Then the trace of K is equal to the integral of its kernel along the diagonal:

$$\mathrm{tr}(K) = \int_0^1 k(x,x)dx.$$

Theorem (Drnovšek, 2017)

Let K be a trace-class kernel operator on $L^2[0,1]$ with a continuous kernel k. Suppose that K has increasing spectrum relative to standard compressions and that the modulus $|K|$ is also a trace-class operator. Then K and $|K|$ are quasinilpotent operators admitting a (common) standard triangularization.

Let K be a trace-class kernel operator on $L^2[0,1]$ with a continuous kernel $k : [0,1] \times [0,1] \to \mathbb{C}$. Then the trace of K is equal to the integral of its kernel along the diagonal:

$$\operatorname{tr}(K) = \int_0^1 k(x,x)dx.$$

Theorem (Drnovšek, 2017)

Let K be a trace-class kernel operator on $L^2[0,1]$ with a continuous kernel k. Suppose that K has increasing spectrum relative to standard compressions and that the modulus $|K|$ is also a trace-class operator. Then K and $|K|$ are quasinilpotent operators admitting a (common) standard triangularization.

Idea of the proof:

Using the continuity of the spectrum for compact operators, one can show that K is quasinilpotent.

For any standard projection P and any n, we have

$$\operatorname{tr}\left((PKP)^n\right) = 0.$$

Then we show that, for any x_1, x_2, ..., x_n in $[0, 1]$, it holds that

$$k(x_1, x_2)k(x_2, x_3)k(x_3, x_4)\cdots k(x_{n-1}, x_n)k(x_n, x_1) = 0.$$

It follows that $\operatorname{tr}(|K|^n) = 0$ for all $n \in \mathbb{N}$, and so $|K|$ is quasinilpotent.

Therefore, $|K|$ admits a standard triangularization, by (a corollary to) de Pagter's theorem.

Idea of the proof:

Using the continuity of the spectrum for compact operators, one can show that K is quasinilpotent.
For any standard projection P and any n, we have

$$\operatorname{tr}\left((PKP)^n\right) = 0.$$

Then we show that, for any x_1, x_2, ..., x_n in $[0,1]$, it holds that

$$k(x_1,x_2)k(x_2,x_3)k(x_3,x_4)\cdots k(x_{n-1},x_n)k(x_n,x_1) = 0.$$

It follows that $\operatorname{tr}(|K|^n) = 0$ for all $n \in \mathbb{N}$, and so $|K|$ is quasinilpotent.
Therefore, $|K|$ admits a standard triangularization, by (a corollary to) de Pagter's theorem.

Idea of the proof:

Using the continuity of the spectrum for compact operators, one can show that K is quasinilpotent.
For any standard projection P and any n, we have

$$\operatorname{tr}\left((PKP)^n\right) = 0.$$

Then we show that, for any x_1, x_2, ..., x_n in $[0,1]$, it holds that

$$k(x_1,x_2)k(x_2,x_3)k(x_3,x_4)\cdots k(x_{n-1},x_n)k(x_n,x_1) = 0.$$

It follows that $\operatorname{tr}(|K|^n) = 0$ for all $n \in \mathbb{N}$, and so $|K|$ is quasinilpotent.
Therefore, $|K|$ admits a standard triangularization, by (a corollary to) de Pagter's theorem.

Idea of the proof:

Using the continuity of the spectrum for compact operators, one can show that K is quasinilpotent.
For any standard projection P and any n, we have

$$\operatorname{tr}((PKP)^n) = 0.$$

Then we show that, for any x_1, x_2, ..., x_n in $[0, 1]$, it holds that

$$k(x_1, x_2)k(x_2, x_3)k(x_3, x_4)\cdots k(x_{n-1}, x_n)k(x_n, x_1) = 0.$$

It follows that $\operatorname{tr}(|K|^n) = 0$ for all $n \in \mathbb{N}$, and so $|K|$ is quasinilpotent.
Therefore, $|K|$ admits a standard triangularization, by (a corollary to) de Pagter's theorem.

Idea of the proof:

Using the continuity of the spectrum for compact operators, one can show that K is quasinilpotent.
For any standard projection P and any n, we have

$$\mathrm{tr}\left((PKP)^n\right) = 0.$$

Then we show that, for any x_1, x_2, ..., x_n in $[0,1]$, it holds that

$$k(x_1,x_2)k(x_2,x_3)k(x_3,x_4)\cdots k(x_{n-1},x_n)k(x_n,x_1) = 0.$$

It follows that $\mathrm{tr}\left(|K|^n\right) = 0$ for all $n \in \mathbb{N}$, and so $|K|$ is quasinilpotent.
Therefore, $|K|$ admits a standard triangularization, by (a corollary to) de Pagter's theorem.

This theorem can be extended to more general measures.

Let $A = \{2, 3, 4, \ldots, N+1\}$ if $N \in \mathbb{N}$, and $A = \mathbb{N} \setminus \{1\}$ if $N = \infty$.

We assume that μ is a Borel measure on $[0, \infty)$ with the support $X = [0, 1] \cup A$, the restriction of μ to $[0, 1]$ is the Lebesgue measure, and $\{j\}$ is an atom of measure 1 for each $j \in A$.

Clearly, the Hilbert space $L^2(X, \mu)$ is the direct sum of $L^2[0, 1]$ and $l^2(A)$.

Let P_C denote the standard projection corresponding to the interval $[0, 1]$, and let P_A denote the standard projection corresponding to the set A.

This theorem can be extended to more general measures.
Let $A = \{2,3,4,\ldots,N+1\}$ if $N \in \mathbb{N}$, and $A = \mathbb{N} \setminus \{1\}$ if $N = \infty$.
We assume that μ is a Borel measure on $[0,\infty)$ with the support $X = [0,1] \cup A$, the restriction of μ to $[0,1]$ is the Lebesgue measure, and $\{j\}$ is an atom of measure 1 for each $j \in A$.
Clearly, the Hilbert space $L^2(X,\mu)$ is the direct sum of $L^2[0,1]$ and $l^2(A)$.
Let P_C denote the standard projection corresponding to the interval $[0,1]$, and let P_A denote the standard projection corresponding to the set A.

This theorem can be extended to more general measures.
Let $A = \{2, 3, 4, \ldots, N+1\}$ if $N \in \mathbb{N}$, and $A = \mathbb{N} \setminus \{1\}$ if $N = \infty$.
We assume that μ is a Borel measure on $[0, \infty)$ with the support $X = [0,1] \cup A$, the restriction of μ to $[0,1]$ is the Lebesgue measure, and $\{j\}$ is an atom of measure 1 for each $j \in A$.
Clearly, the Hilbert space $L^2(X, \mu)$ is the direct sum of $L^2[0,1]$ and $l^2(A)$.
Let P_C denote the standard projection corresponding to the interval $[0,1]$, and let P_A denote the standard projection corresponding to the set A.

This theorem can be extended to more general measures.
Let $A = \{2, 3, 4, \dots, N+1\}$ if $N \in \mathbb{N}$, and $A = \mathbb{N} \setminus \{1\}$ if $N = \infty$.
We assume that μ is a Borel measure on $[0, \infty)$ with the support $X = [0,1] \cup A$, the restriction of μ to $[0,1]$ is the Lebesgue measure, and $\{j\}$ is an atom of measure 1 for each $j \in A$.
Clearly, the Hilbert space $L^2(X, \mu)$ is the direct sum of $L^2[0,1]$ and $l^2(A)$.
Let P_C denote the standard projection corresponding to the interval $[0,1]$, and let P_A denote the standard projection corresponding to the set A.

This theorem can be extended to more general measures.
Let $A = \{2, 3, 4, \dots, N+1\}$ if $N \in \mathbb{N}$, and $A = \mathbb{N} \setminus \{1\}$ if $N = \infty$.
We assume that μ is a Borel measure on $[0, \infty)$ with the support $X = [0, 1] \cup A$, the restriction of μ to $[0, 1]$ is the Lebesgue measure, and $\{j\}$ is an atom of measure 1 for each $j \in A$.
Clearly, the Hilbert space $L^2(X, \mu)$ is the direct sum of $L^2[0, 1]$ and $l^2(A)$.
Let P_C denote the standard projection corresponding to the interval $[0, 1]$, and let P_A denote the standard projection corresponding to the set A.

Theorem (Drnovšek, 2017)

Let K be a trace-class kernel operator on $L^2(X,\mu)$ with a continuous kernel k. Suppose that K has increasing spectrum relative to standard compressions and that its modulus $|K|$ is also a trace-class operator. Then K and $|K|$ admit a (common) standard triangularization. Furthermore, the operators K and $P_A K P_A$ have the same non-zero eigenvalues with the same algebraic multiplicities. This holds also for the operators $|K|$ and $P_A|K|P_A$, while the operators $P_C K P_C$ and $P_C|K|P_C$ are both quasinilpotent.

Theorem (Drnovšek, 2017)

Let K be a trace-class kernel operator on $L^2(X, \mu)$ with a continuous kernel k. Suppose that K has increasing spectrum relative to standard compressions and that its modulus $|K|$ is also a trace-class operator. Then K and $|K|$ admit a (common) standard triangularization. Furthermore, the operators K and $P_A K P_A$ have the same non-zero eigenvalues with the same algebraic multiplicities. This holds also for the operators $|K|$ and $P_A|K|P_A$, while the operators $P_C K P_C$ and $P_C|K|P_C$ are both quasinilpotent.

Theorem (Drnovšek, 2017)

Let K be an operator on $L^2(X,\mu)$ of rank $n \in \mathbb{N}$. If K has increasing spectrum, then it admits a standard triangularization. Furthermore, there exist a positive integer $m \le 2n+1$ and a partition $\{E_1,\dots,E_m\}$ of X such that, relative to the decomposition $L^2(X,\mu) = \bigoplus_{j=1}^m L^2(E_j,\mu|_{E_j})$, K has the form

$$K = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & \dots & K_{1,m-1} & K_{1,m} \\ 0 & K_{2,2} & K_{2,3} & K_{2,4} & \dots & K_{2,m-1} & K_{2,m} \\ 0 & 0 & K_{3,3} & K_{3,4} & \dots & K_{3,m-1} & K_{3,m} \\ 0 & 0 & 0 & 0 & \ddots & K_{4,m-1} & K_{4,m} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & K_{m-1,m-1} & K_{m-1,m} \\ 0 & 0 & 0 & 0 & \dots & 0 & K_{m,m} \end{bmatrix},$$

where each diagonal block $K_{j,j}$ can be non-zero only when $L^2(E_j,\mu|_{E_j})$ is a one-dimensional space (corresponding to an atom), and in this case $K_{j,j}$ is a non-zero eigenvalue of K.

Theorem (Drnovšek, 2017)

Let K be an operator on $L^2(X,\mu)$ of rank $n\in\mathbb{N}$. If K has increasing spectrum, then it admits a standard triangularization. Furthermore, there exist a positive integer $m\le 2n+1$ and a partition $\{E_1,\dots,E_m\}$ of X such that, relative to the decomposition $L^2(X,\mu)=\bigoplus_{j=1}^m L^2(E_j,\mu|_{E_j})$, K has the form

$$K=\begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & \dots & K_{1,m-1} & K_{1,m} \\ 0 & K_{2,2} & K_{2,3} & K_{2,4} & \dots & K_{2,m-1} & K_{2,m} \\ 0 & 0 & K_{3,3} & K_{3,4} & \dots & K_{3,m-1} & K_{3,m} \\ 0 & 0 & 0 & 0 & \ddots & K_{4,m-1} & K_{4,m} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & K_{m-1,m-1} & K_{m-1,m} \\ 0 & 0 & 0 & 0 & \dots & 0 & K_{m,m} \end{bmatrix},$$

where each diagonal block $K_{j,j}$ can be non-zero only when $L^2(E_j,\mu|_{E_j})$ is a one-dimensional space (corresponding to an atom), and in this case $K_{j,j}$ is a non-zero eigenvalue of K.

The bound $2n+1$ in the last theorem cannot be improved.

Example

Let $n \in \mathbb{N}$, and let e_1, e_2, …, e_{2n+1} be the standard basis vectors of $\mathbb{C}^{2n+1}$. For each $j = 1, 2, \ldots, n$, let $f_j = \sum_{i=2j}^{2n+1} e_i$. Define

$$K = \sum_{j=1}^{n} (e_{2j-1} + e_{2j}) \cdot f_j^t.$$

For example, if $n = 2$ then

$$K = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then K is an upper triangular matrix of rank n, and it has increasing spectrum. Furthermore, it already has the form guaranteed by Theorem and we cannot decrease the number of diagonal blocks.

Bibliography

1. L. W. Marcoux, M. Mastnak, H. Radjavi, *Triangularizability of operators with increasing spectrum*, J. Funct. Anal. 257 (2009), 3517–3540.
2. R. Drnovšek, *Triangularizability of trace-class operators with increasing spectrum*, J. Math. Anal. Appl. 447 (2017), 1102–1115.